An Assessment of Renormalization Methods in the Statistical Theory of Isotropic Turbulence

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In the name of God; at the beginning and the end.

“The job is to ask questions - it always was - and to ask them as inexorably as I can. And to face the absence of precise answers with a certain humility.”

- Arthur Miller

Abstract

For the latter half of the last century renormalization methods have played an important part in tackling problems in fundamental physics and in providing a deeper understanding of systems with many interacting scales or degrees of freedom with strong coupling. The study of turbulence is no exception, and this thesis presents an investigation of renormalization techniques available in the study of the statistical theory of homogeneous and isotropic turbulence.

The thesis consists of two parts which assess the two main renormalization approaches available in modeling turbulence. In particular we will be focusing on the renormalization procedures developed by McComb and others.

The first part of this thesis will discuss Renormalization Group (RG) approaches to turbulence, with a focus on applications to reduce the degrees of freedom in a large-eddy simulation. The RG methods as applied to classical dynamical systems will be reviewed in the context of the Navier-Stokes equations describing fluid flow. This will be followed by introducing a functional based formalism of a conditional average first introduced by McComb, Roberts and Watt [Phys. Rev A 45, 3507 (1992)] as a tool for averaging out degrees of freedom needed in an RG calculation. This conditional average is then used in a formal RG calculation applied to the Navier-Stokes equations, originally done by McComb and Watt [Phys. Rev. A 46, 4797 (1992)], and later revised by McComb and Johnston [Physica A 292, 346 (2001)]. A correction to the summing of the time-integral detailed in the latter work is shown to introduce an extra viscous life-time term to the denominator of the increment to the renormalized viscosity and is shown to have a negligible effect in the numerical calculations. We follow this study by outlining some problems with the previous approach. In particular it is shown that a cross-term representing the interaction between high and low wavenumber modes which was neglected in the previous studies on the grounds that it does not contribute to energy dissipation, does in fact

contribute significantly. A heuristic method is then put forward to include the effects of this term in the RG calculation. This leads to results which agree qualitatively with numerical calculations of eddy-viscosities. We finish this part of the thesis with an application of the RG method to the modeling of a passive scalar advected by a turbulent velocity field.

The second part of this thesis will begin by reviewing Eulerian renormalized perturbation theory attempts in closing the infinite moment hierarchy introduced by averaging the Navier-Stokes equations. This is followed by presenting a new formulation of the local energy transfer theory (LET) of McComb *et. al.* [J. Fluid Mech. 245, 279 (1992)] which resolves some problems of previous derivations. In particular we show by the introduction of time-ordering that some previous problems with the exponential representation of the correlator can be overcome. Furthermore, we show that the singularity in the LET propagator equation cancels by way of a counter-term. We end this study by introducing a single-time Markovian closure based on LET which, unlike other Markovian closures, does not rely on any arbitrary parameters being introduced in the theory.

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Lastly, I would like to thank my family and especially my parents, for whom this thesis is dedicated.

Declarations

I declare that this thesis was composed by myself and that, except where explicitly stated otherwise in the text, the work contained therein is my own or was carried out in collaboration with Professor W. D. McComb.

The code used to program and compute the RG iteration was originally written by C. Johnston and was modified by myself to obtain the results in Chapters 4–6. The computation in Chapter 6 was carried out with the help of A. Kuczaj from the University of Twente, The Netherlands.

Aspects of the work in chapters 4, 5 and 6 were presented at the 10th European Turbulence Conference, Trondheim, Norway, June 29 - July 3, 2004, and are contained in the proceedings of this conference.

Some of the work in Chapter 8 has been published in the APS Physical Review E journal [PRE **70**, 066303 (2004)] and is reproduced in Appendix D.

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Contents

Chapter 1

Introduction

‘Turbulence was a problem with pedigree’ starts James Gleick in his most acclaimed book ‘Chaos’[1]. There seems to exist no lack of sensationalism when talking about this most familiar and ubiquitous of non-linear phenomena. In this case, Gleick is referring to the generations of great mathematicians and physicists who have bent their minds on turbulence. In the field of chaos theory where systems with many degrees of freedom are studied, turbulence is acknowledged as the archetypal form; the mother of chaotic phenomena. When asked what turbulence physically is, one may be confronted by the loose labeling of the unstable, irregular, disordered and random motion of fluids; and yet after all this pessimism, it is not quite. There is a strange and eerie lure to the silent beauty of the vortical motion of smoke as it rises from a stick of incense; a streamlined or *laminar* flow develops instabilities beyond which a cascade of vortices usher in the rule of turbulence. Just under all the apparent randomness that beguiles us, there is an order or coherence that keeps us entranced. Unlike many other areas in physics where the systems under study are beyond the extremes of human resolution and are thus rather esoteric and fantastical, turbulence is a mesoscopic phenomenon i.e. it is seen in every-day phenomena and at most scales. Thus it is no exageration to say that most problems in science and engineering have the problem of turbulence cropping up somewhere within their respective studies. And this is where the great intrigue lies. It is seen everywhere, it is easy to produce in the lab and we even claim to have the equations of motion for it; yet it is a rather discouraging realisation to find that we do not know much about turbulence. Turbulence is a multi-scale phenomenon, but its intractability has forced its practitioners to

compartmentalise themselves to a particular scale or set of scales which don't seem to 'talk' to each other. This unfortunate scenario means that the field of turbulence research is couched in various different formalisms and appears in a variety of guises, all depending on what aspect one is studying. Although the world of turbulence research is multi-faceted, this humble introduction aims to provide a coherent background on the formalism and tools needed to study one of these aspects of which the subject matter of this thesis is concerned. This being the statistical study of fluid turbulence.

1.1 The dynamical equations of incompressible fluid flow

In 1751 Euler formulated the equations for inviscid fluid flow. In 1823 Navier added the effects of viscosity, and so were born the Navier-Stokes equations (NSE) which express the conservation of momentum (Newton's second law) in fluid flow

$$\frac{\partial}{\partial t}U_\alpha(\mathbf{x},t)+U_\beta(\mathbf{x},t)\frac{\partial}{\partial x_\beta}U_\alpha(\mathbf{x},t)=-\frac{1}{\rho}\frac{\partial}{\partial x_\alpha}P(\mathbf{x},t)+\nu\frac{\partial^2}{\partial x_\beta\partial x_\beta}U_\alpha(\mathbf{x},t)+f_\alpha(\mathbf{x},t), \tag{1.1}$$

where $U_\alpha(\mathbf{x},t)$ is the fluid velocity field at position $\mathbf{x}$ and time t, $P(\mathbf{x},t)$ is the pressure field in the fluid, ρ is the density, ν is the kinematic viscosity and $f_\alpha(\mathbf{x},t)$ is the external force per unit mass doing work on the system. In this thesis we will only be concerned with flows of incompressible fluids i.e. those flows in which the density of the fluid is constant in both space and time. When this requirement is imposed upon the equation which expresses mass conservation, we obtain the continuity condition

$$\frac{\partial}{\partial x_\beta}U_\beta(\mathbf{x},t)=0. \tag{1.2}$$

In as much as equations (1.1) and (1.2) govern the behaviour of the flow of an incompressible fluid, and thus their applications to turbulence, their application is strictly to mesoscale phenomena. They are constructed with a view of the fluid as a continuum and thus are not applicable at very small scales compara-

ble to the mean free path of a gas-like particle [2] where a molecular picture is needed to provide a detailed description. In the NSE (1.1), the connection with the molecular details is provided by the viscosity, ν, which in essence represents a mean effect of the molecular interactions that macroscopically manifest themselves as the 'stickiness' of the fluid. More importantly it must be stated (although it might seem quite obvious) that the NSE shows that turbulence is not of a molecular origin but of some intrinsic property of fluid *flow*. This intrinsic property of the flow manifests itself in the quadratic non-linearity of the NSE which indicates the collective nature of the flow i.e. the flow interferes and interacts with itself[1] so as to produce instabilities which develop to produce what we call turbulence. We will also find that the continuity condition allows one to write the pressure term in the NSE as a non-linearity. This identification of the source of turbulence in the NSE will be needed later in the development of renormalized perturbation theories where one needs the ability to switch off the non-linearity.

1.1.1 Self-similarity and scaling

Amongst the most powerful tools in the arsenal of a student of turbulence is dimensional analysis. In any physical study, one aims to obtain relationships amongst the quantities that characterise the phenomena being studied. In circumstances where the relevant parameters and variables are known, a sensible application of dimensional analysis can often be used to characterise a phenomena or even suggest relationships in the form of functional equations, and in particular generalised homogeneous functions, between the governing parameters of a system (see Barenblatt [4]). Several examples of this exist in the study of fluid flow, for which the most well known are the von Karman-Prandtl universal logarithmic law for velocity distributions and the Kolmogorov form for the universal inertial range in the spectrum for the turbulent kinetic energy (see later). However, the most important result of dimensional analysis in the study of fluid flows is probably the Reynolds number.

Although we have known the NSE for nearly two centuries it is widely acknowledged that the scientific study of fluid flows began with the work of Osborne

[1] The quadratic non-linearity can be seen to be analagous to the ϕ^4 self-interaction term in the Ginzburg-Landau Hamiltonian of scalar field theory [3].

Reynolds (1883) [5] and his experiments with pipe flows. Fluid flows are known to exhibit *self-similarity* and *scaling* phenomena. This means that as long as the geometry of the experiment/environment remains the same, then the geometry/form of the resultant flow will also be invariant. For example a pipe could have various diameters, but the form of the velocity distribution will stay the same. This fact can be illustrated more dramatically by comparing the mushroom-like surface generated by a water drop falling into a pool of clear water and the aftermath of a nuclear test [6].

By scaling the relevant dependent variables in the study so as to make them dimensionless i.e. independent of the units used to measure, Reynolds made the data from different experiments in pipe flows all collapse onto a single curve. The primary dimensionless parameter which was used to do this was

$$Re = \frac{UL}{\nu}, \tag{1.3}$$

where U is the characteristic velocity of the flow, L is the characteristic length and ν is the kinematic viscosity. For pipe flow $L = D$ the diameter of the pipe and U is the bulk flow of the fluid at the centre of the pipe. Following a proposal by the physicist A. Sommerfield, Re was named the Reynolds number in honour of its originator [4].

The Reynolds number is a parameter used to characterise how turbulent a flow is. At 'low' Re, flows can be considered laminar whilst beyond a certain Re the flow becomes unstable and turbulent. For example, in pipe flow with the Re defined as above, the transition to turbulence occurs at $Re \sim 2000$ [5]. This fact allows one to get to (1.3) in a quick way by realising that the onset of turbulence is due to the competing influence of the non-linear and viscous terms in the NSE. Thus by applying dimensional analysis to the ratio of these two quantities we get

$$\frac{[non-linear\,term]}{\nu[viscous\,term/\nu]} = \frac{U^2L^{-1}}{\nu UL^{-2}} = \frac{UL}{\nu}, \tag{1.4}$$

where the square brackets represent the operation of getting the dimensions of the argument.

The difficult matter in assigning the Reynolds number to a particular problem lies with choosing a characteristic velocity and length scale.

1.2 The turbulence problem in physics

From the introduction of the Reynolds number one might come to think that the great old problem of turbulence is primarily concerned with being able to predict the transition from laminar to turbulent flow. However, this is just one of the turbulence problems and the nearest study of this, which is known as *stability theory*, would probably be in engineering and is a subset of control theory.

Physicists want to study the physics of the phenomena, i.e. the fundamental nature of, and what is inherent in all universal turbulence phenomena. In this respect we aim to study the characteristics of turbulence phenomena away from any external boundaries and independent from initial conditions. We have already seen that turbulence exhibits self-similar behaviour and thus we anticipate that turbulence has unique properties inherent in itself. We have also seen that a direct solution of the NSE is not considered practical due to the irregular nature of turbulence. Thus the physicist's turbulence problem may be stated as being able

> to provide a statistical description of turbulence, from which predictions may then be made on mean quantities of interest such as velocities and pressures, for well developed turbulent flows.

1.2.1 Homogeneous isotropic turbulence

In order to study the physics of turbulence, where we seek to obtain the universal properties, one desires to make as many simplifications as possible so as to make the problem easier to handle. One needs to concentrate on the simplest non-trivial problems. It is widely agreed that the assumptions of homogeneity and isotropy present the simplest non-trivial case whilst retaining all the essential physics of turbulence. Accordingly, in this thesis we will restrict our study to the case of homogeneous and isotropic turbulence with zero mean flow (see later). The mathematical simplifications resulting from this will be shown in the next section.

Before we end this section, a brief mention should also be made about the mathematicians turbulence problem. This problem involves the proof of the

existence and smoothness of solutions for the NSE . Throughout this thesis, and in fact amongst the physics community, the NSE is assumed to represent all of turbulence and thus we will be implicitly assuming that solutions do exist. Moreover we will assume that the solutions to the NSE are unique i.e. unique initial conditions imply a unique solution.

1.3 Statistical theory

The irregularity or random character of all turbulent flows makes a deterministic approach to turbulence problems very hard indeed. Thus, we have to resort to statistical methods. But how can one treat something which, from the NSE is inherently deterministic, as probabilistic? What constitutes the ensemble? Batchelor [2] identifies the ensemble to be comprised of the velocity realisations which are generated for many identically repeated experiments. This implicitly assumes that the velocity field is a random function of $\mathbf{x}$ and t, and accordingly is distributed according to some probability laws. We will return to this point later in this section and in Chapter 3. However, now that the ensemble is loosely defined, we may take mean averages as is normally done in statistical physics.

1.3.1 Fourier space decomposition

In order to make manifest the many scales or degrees of freedom (DOF's) involved in turbulence, we transform the NSE to Fourier space where the DOF's represented by the wavenumber modes make up the basis. This way the turbulence problem can easily be seen to be another facet of many-body physics. In this thesis we will primarily be interested in the spatial degrees of freedom and will thus be needing the spatial Fourier transform pair defined by

$$U_\alpha(\mathbf{x},t) = \int d^3k U_\alpha(\mathbf{k},t) e^{i\mathbf{k}\cdot\mathbf{x}} \tag{1.5}$$

and

$$U_\alpha(\mathbf{k},t) = \left(\frac{1}{2\pi}\right)^3 \int d^3x U_\alpha(\mathbf{x},t) e^{-i\mathbf{k}\cdot\mathbf{x}}. \tag{1.6}$$

Transforming to Fourier space has the dual advantages of being able to talk about the coupling between the degrees of freedom or Fourier modes, and also of turning differential operators into products. Fourier transforming the NSE (1.1) and the continuity condition (1.2) we obtain

$$\left[\frac{\partial}{\partial t}+\nu k^2\right] U_\alpha(\mathbf{k},t) = -ik_\beta \int d^3j U_\alpha(\mathbf{j},t)U_\beta(\mathbf{k}-\mathbf{j},t) - k_\alpha P(\mathbf{k},t) + f_\alpha(\mathbf{k},t) \quad (1.7)$$

and

$$k_\alpha U_\alpha(\mathbf{k},t) = 0 \quad (1.8)$$

where we have chosen $\rho = 1$. From (1.7), we can see that the quadratic non-linearity has now become a convolution over all the spatial modes. More on Fourier transforming the NSE can be found in McComb [5].

As mentioned earlier we will now eliminate the pressure. We will do this by multiplying (1.7) by k_α and using the continuity condition (1.8) to get

$$P(\mathbf{k},t) = \frac{-ik_\alpha k_\beta}{k^2} \int d^3j U_\alpha(\mathbf{j},t)U_\beta(\mathbf{k}-\mathbf{j},t), \quad (1.9)$$

and where in this procedure we have to require the force to be divergence-free $k_\alpha f_\alpha(\mathbf{k},t) = 0$. Substituting (1.9) into (1.7) and with some manipulation [5] we obtain the solenoidal or divergence-free NSE in Fourier space

$$\left[\frac{\partial}{\partial t}+\nu k^2\right] U_\alpha(\mathbf{k},t) = M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j U_\beta(\mathbf{j},t)U_\gamma(\mathbf{k}-\mathbf{j},t) + f_\alpha(\mathbf{k},t), \quad (1.10)$$

where

$$M_{\alpha\beta\gamma}(\mathbf{k}) = \frac{1}{2i}\left[k_\beta P_{\alpha\gamma}(\mathbf{k}) + k_\gamma P_{\alpha\beta}(\mathbf{k})\right] \quad (1.11)$$

and

$$P_{\alpha\beta}(\mathbf{k}) = \delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}. \quad (1.12)$$

Note that $M_{\alpha\beta\gamma}(\mathbf{k})$ remains invariant under a swapping of the indices β and γ, and (1.10) is invariant under interchange of the dummy arguments $\mathbf{j}$ and $\mathbf{k}-\mathbf{j}$ in the integrand; these symmetries will be needed later.

Next we will follow the procedure taken by Reynolds of decomposing the velocity field into the sum of the mean $\langle U_\alpha(\mathbf{k},t)\rangle$ and fluctuation $u_\alpha(\mathbf{k},t)$ parts

$$U_\alpha(\mathbf{k},t) = \langle U_\alpha(\mathbf{k},t)\rangle + u_\alpha(\mathbf{k},t), \quad (1.13)$$

where the procedure of taking an average is represented by the angular brackets $\langle\,\rangle$, and we are concerned with ensemble averages (see chapter 3). An immediate result from this is that $\langle u_\alpha(\mathbf{k},t)\rangle = 0$.

In this thesis we are working with flows with zero mean velocity, therefore we can replace $U_\alpha(\mathbf{k},t)$ in (1.10) and (1.8) with $u_\alpha(\mathbf{k},t)$. These equations will be the ones refered to as the NSE from henceforth. It is important to realise that the Reynolds decomposition is in fact just partitioning the flow into the smooth or laminar parts, and those which represent the random/fluctuating or turbulent parts. Thus, it is clear that working with a flow of zero mean velocity implies that the flow concerned is entirely turbulent. It is also slightly artificial as turbulence is a property of flows, and it is the flow which sustains the turbulence (see McComb [5] section 1.3). This conceptual problem is circumvented by our force term in the NSE, which can be seen to be sustaining the turbulence as a replacement of the flow. One should also mention that the concept of isotropic turbulence is, intuitively, very much connected to zero mean flows, as a mean flow will always introduce some anisotropies in the motion of the fluid. The possible strategies and difficulties involved in generating homogeneous and isotropic zero mean flow can be found in McComb [5] and the recent study of Hwang and Eaton [7].

Before we move on, we can now use the Reynolds decomposition to attempt to say something regarding the nature of the ensemble discussed earlier as being comprised of velocity realisations generated from many identical experiments. In the context of the NSE this does not make sense, as identical experiments mean identical initial conditions which imply identical velocity realisations; and an ensemble consisting of just one realisation is not really an ensemble. Instead, what we can say is that the ensemble of *fluctuating* velocities can be seen to be comprised of all velocity fluctuation realisations, and thus velocities (pre-Reynolds decomposition), which correspond to a particular mean velocity. This approach for defining the ensemble seems to be coherent with the random nature of turbulence and the adoption of the NSE as describing turbulent flows. Note that we can also get around this problem by using random initial conditions.

1.3.2 Velocity moments

The two-point, two-time velocity moment is given by

$$\langle u_\alpha(\mathbf{k},t)u_\beta(\mathbf{k}',t')\rangle \quad =: \quad Q_{\alpha\beta}\left(\mathbf{k},\mathbf{k}';t,t'\right), \tag{1.14}$$

and is what is known as a second order moment. Higher order n-point, m-time velocity moments are written in a similar way. However, the second order moment will be the most important moment studied in this thesis. It is also known as the covariance or correlation tensor.

As the primary concern of this thesis is with homogeneous and isotropic turbulence, it would be instructive to see what effects these symmetries have on our system. Homogeneity principally implies that we can write (1.14) as

$$\langle u_\alpha(\mathbf{k},t)u_\beta(\mathbf{k}',t')\rangle = Q_{\alpha\beta}\left(\mathbf{k}';t,t'\right)\delta(\mathbf{k}+\mathbf{k}'). \tag{1.15}$$

Thus homogeneity has essentially rendered a function of the two variables $\mathbf{k}$ and $\mathbf{k}'$ into a function of a single variable $\mathbf{k}'$. When (1.15) is integrated over $\mathbf{k}'$ we get

$$Q_{\alpha\beta}\left(\mathbf{k};t,t'\right) = \int d^3k' \left\langle u_\alpha(\mathbf{k},t)u_\beta(\mathbf{k}',t')\right\rangle. \tag{1.16}$$

On a first glance, isotropy implies

$$Q_{\alpha\beta}\left(\mathbf{k};t,t'\right) = Q_{\alpha\beta}\left(-\mathbf{k};t,t'\right)\ , \tag{1.17}$$

and rotational invariance such that

$$Q_{\alpha\beta}\left(\mathbf{k};t,t'\right) = Q_{\beta\alpha}\left(\mathbf{k};t,t'\right)\ ; \tag{1.18}$$

however, it also allows one to reduce the nine scalar functions of the components of the above tensor to just one. This was first done by Robertson [8] by using the fact that an isotropic tensor $A_{\alpha\beta}(\mathbf{k})$, say, can be expressed in terms of the invariants of the rotation group such that one can write

$$A_{\alpha\beta}(\mathbf{k}) = P_{\alpha\beta}(\mathbf{k})A(k), \tag{1.19}$$

where $P_{\alpha\beta}(\mathbf{k})$ was given earlier in (1.12). So in total, homogeneity and isotropy

allows one to write the original $\langle u_\alpha(\mathbf{k},t)u_\beta(\mathbf{k}',t')\rangle$ as

$$\langle u_\alpha(\mathbf{k},t)u_\beta(\mathbf{k}',t')\rangle = \delta(\mathbf{k}+\mathbf{k}')P_{\alpha\beta}(\mathbf{k})Q(k;t,t'). \tag{1.20}$$

Later on, we will also encounter the concept of statistical stationarity, which means that there exists no absolute time frame from which we measure, so that we may write

$$Q_{\alpha\beta}\left(\mathbf{k};t,t'\right) = Q_{\alpha\beta}\left(\mathbf{k};t-t'\right). \tag{1.21}$$

In this respect we may consider stationarity to be just temporal homogeneity.

1.3.3 The isotropic spectral energy density

The turbulence kinetic energy per unit mass is given by

$$\begin{aligned} E(t) &= \int dk E(k,t) \\ &= \frac{3}{2}\left\langle u(t)^2\right\rangle. \end{aligned} \tag{1.22}$$

This can be written as

$$\begin{aligned} E(t) &= \frac{3}{2}\left\langle u(t)^2\right\rangle \\ &= \frac{1}{2}\int d^3k \mathrm{tr}\left\langle u_\alpha(\mathbf{k},t)u_\beta(\mathbf{k},t)\right\rangle \\ &= \frac{1}{2}\int d^3k \mathrm{tr} P_{\alpha\beta}(\mathbf{k})Q(k;t) \\ &= \int d^3k Q(k;t), \end{aligned} \tag{1.23}$$

where we have used (1.12) to see that $\mathrm{tr}P_{\alpha\beta}(\mathbf{k}) = 2$. Doing the angular integration in (1.23) results in

$$\int d^3k Q(k;t) = \int dk 4\pi k^2 Q(k;t), \tag{1.24}$$

which when compared with (1.22) allows us to obtain the equality

$$E(k,t) = 4\pi k^2 Q(k;t), \tag{1.25}$$

from which we make the interpretation of $Q(k,t)$ as the spectral energy density.

A detailed description on the ideas stated here and how to arrive at these relations can be found in McComb [5] and also in the review by Rose and Sulem [9] where the analogies between turbulence theory and statistical physics are explored.

1.3.4 The closure problem

Equation (1.10) is a vector equation i.e. it consists of three equations, for a vector field of three components. Thus, we have three equations for three unknowns (the viscosity, density and force being assumed to be prescribed). Because the number of unknowns equals the number of equations, the system is said to be 'closed' and in principle should provide solutions. However, as illustrated below, to date this has not proven to be the case.

We write (1.10) in a shorthand notation as

$$\widehat{L}U = \widehat{M}UU + f, \tag{1.26}$$

where $\widehat{L}$ and $\widehat{M}$ are linear and convolution (non-linear) operators respectively, whose forms can be ascertained from (1.10). Ensemble averaging (1.26) we obtain

$$\widehat{L}\left\langle U\right\rangle = \widehat{M}\left\langle UU\right\rangle + \left\langle f\right\rangle, \tag{1.27}$$

which shows that the solution for $\langle U\rangle$ depends upon the second-order moment $\langle UU\rangle$. However, we do not know $\langle UU\rangle$, and thus constructing an equation for $\langle UU\rangle$ by multiplying (1.26) by U and then averaging we obtain

$$\widehat{L}\left\langle UU\right\rangle = \widehat{M}\left\langle UUU\right\rangle + \left\langle f\right\rangle, \tag{1.28}$$

which shows that the solution for $\langle UU\rangle$ depends upon the third-order moment $\langle UUU\rangle$. We can carry this procedure on to generate what is known as the moment hierarchy i.e. the n^{th} order moment depends upon the $n+1^{th}$ order moment. So beginning with a system of n equations for n unknowns, we have, due to our desire to obtain mean quantities, ended up with a set of n equations for $n+1$ moments (the unknowns). Closing this hierarchy is what is known as the 'moment closure problem' and is primarily, implicitly or explicitly, what the turbulence problem is concerned with.

1.4 Kinetics of incompressible fluid flow

1.4.1 Modal energy transport

From the NSE we can derive an equation for energy transport amongst the Fourier modes. This is obtained by first multiplying the NSE for $u_\alpha(\mathbf{k},t)$ by $u_\sigma(-\mathbf{k},t)$, and then multiplying the NSE for $u_\sigma(-\mathbf{k},t)$ by $u_\alpha(\mathbf{k},t)$. Adding the resulting two equations and averaging we obtain

$$\begin{aligned}\left[\frac{\partial}{\partial t}+2\nu k^2\right]\langle u_\sigma(-\mathbf{k},t)u_\alpha(\mathbf{k},t)\rangle = M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\,\langle u_\sigma\left(-\mathbf{k},t\right)u_\beta(\mathbf{j},t)u_\gamma(\mathbf{k}-\mathbf{j},t)\rangle \\ +M_{\sigma\beta\gamma}(-\mathbf{k})\int d^3j\,\langle u_\alpha\left(\mathbf{k},t\right)u_\beta(-\mathbf{j},t)u_\gamma(-\mathbf{k}+\mathbf{j},t)\rangle \\ +\langle u_\sigma(-\mathbf{k},t)f_\alpha(\mathbf{k},t)\rangle+\langle f_\sigma(-\mathbf{k},t)u_\alpha(\mathbf{k},t)\rangle\,. \qquad (1.29)\end{aligned}$$

Making a substitution using (1.20), taking the trace of the entire expression, and then substituting (1.25), we get the **energy balance equation** representing the turbulent kinetic energy transport/exchange between Fourier modes

$$\left[\frac{\partial}{\partial t}+2\nu k^2\right]E(k,t)=T(k,t)+W(k,t), \qquad (1.30)$$

where

$$\begin{aligned}T(k,t) &= 2\pi k^2 M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\,\{\langle u_\alpha\left(-\mathbf{k},t\right)u_\beta(\mathbf{j},t)u_\gamma(\mathbf{k}-\mathbf{j},t)\rangle \\ &\quad -\langle u_\alpha\left(\mathbf{k},t\right)u_\beta(-\mathbf{j},t)u_\gamma(-\mathbf{k}+\mathbf{j},t)\rangle\} \qquad (1.31)\end{aligned}$$

is the non-linear transfer term, $W(k,t)$ is the rate of input energy term due to forcing and $2\nu k^2E(k,t)$ is the dissipation term. A schematic of the behaviour of these terms is shown in figure 1.1.

Before we move on, an important result one can ascertain from the energy balance equation is that the pressure has no role in transferring energy between different modes and thus different scales. This can be seen by expanding the $M_{\alpha\beta\gamma}(\mathbf{k})$ projector to reveal the $P_{\alpha\gamma}(\mathbf{k})$ and then expanding this to show the $k_\alpha k_\gamma/k^2$ term, that has its origin in the pressure term in the NSE which was eliminated earlier using the incompressibility condition. (1.31) involves products of this term with $u_\alpha\left(\mathbf{k},t\right)$ which in particular results in $k_\alpha u_\alpha\left(\mathbf{k},t\right)=0$ due to the incompressibility condition. Batchelor ([2] section 5.2 and 5.5) shows

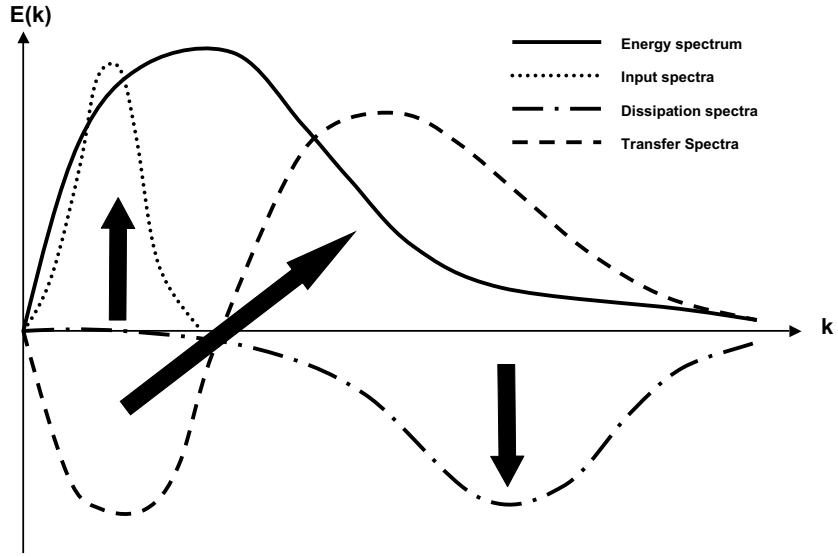

Figure 1.1: Schematic showing the behaviour of the terms in (1.30), and their respective contribution to the energy spectra for a particular value of t.

that the pressure is responsible for transferring energy to different components of the three dimensional velocity field, and is thus responsible for a tendency to isotropy in the equations of motion.

Another important relation to note is the Detailed energy balance identity (see Appendix B)

$$\tilde{T}(-k|j,l) + \tilde{T}(l|-k,j) + \tilde{T}(j|l,-k) = 0 \ , \tag{1.32}$$

where

$$\tilde{T}(-k|j,l) = M_{\alpha\beta\gamma}(\mathbf{k}) \left[Q_{\alpha\beta\gamma}(-\mathbf{k},\mathbf{j},\mathbf{l};t) - Q_{\alpha\beta\gamma}(\mathbf{k},-\mathbf{j},-\mathbf{l};t) \right] \ , \tag{1.33}$$

and

$$Q_{\alpha\beta\gamma}(-\mathbf{k},\mathbf{j},\mathbf{k}-\mathbf{j};t) = \langle u_\alpha(-\mathbf{k},t) u_\beta(\mathbf{j},t) u_\gamma(\mathbf{k}-\mathbf{j},t) \rangle \ . \tag{1.34}$$

The Detailed energy balance identity relates different permutations of the energy transfer triad interactions.

1.4.2 Energy flux through the system

Equation (1.30) tells us about the energy balance at each mode k. We can obtain the total energy flux in the entire system for each of the terms in (1.30)

by integrating (1.30) over all k-space

$$\begin{aligned}\frac{\partial}{\partial t}E(t) &= -\int dk 2\nu k^2 E(k,t) + \int dk W(k,t) \\ &\quad + \int dk T(k,t). \end{aligned} \tag{1.35}$$

This represents the rate at which energy is entering and leaving our system. $E(t)$ represents the total energy contained within the system at time t,

$$\int dk W(k,t) = \varepsilon_w(t) \tag{1.36}$$

is the total rate of energy being inputed into the system i.e. the total work being done by the forcing term $f_\alpha(\mathbf{k},t)$, and

$$\int dk 2\nu k^2 E(k,t) = \varepsilon_d(t) \tag{1.37}$$

represents the total rate at which energy is being lost from the system due to viscous dissipation i.e. molecular friction resulting in heat. The last term on the RHS of equation (1.35), which represents the action of the nonlinear term, is conservative

$$\int dk T(k,t) = 0 \tag{1.38}$$

i.e. it does no work on the system as a whole. The proof of this is shown in Appendix B. Equation (1.35) can thus be written as

$$\frac{\partial}{\partial t}E(t) = \varepsilon_w(t) - \varepsilon_d(t), \tag{1.39}$$

so that for a steady state where $\frac{\partial}{\partial t}E(t) = 0$ we have the dissipation rate becoming a constant

$$\varepsilon_w = \varepsilon_d = \varepsilon. \tag{1.40}$$

1.4.3 The Richardson Cascade

As mentioned earlier, for sufficiently large Reynolds numbers, practically every three-dimensional hydrodynamic flow becomes turbulent. The development of this turbulence is essentially vortical and, when viewed in most flows, involves the production of vortices or eddies of smaller and smaller scale each inter-

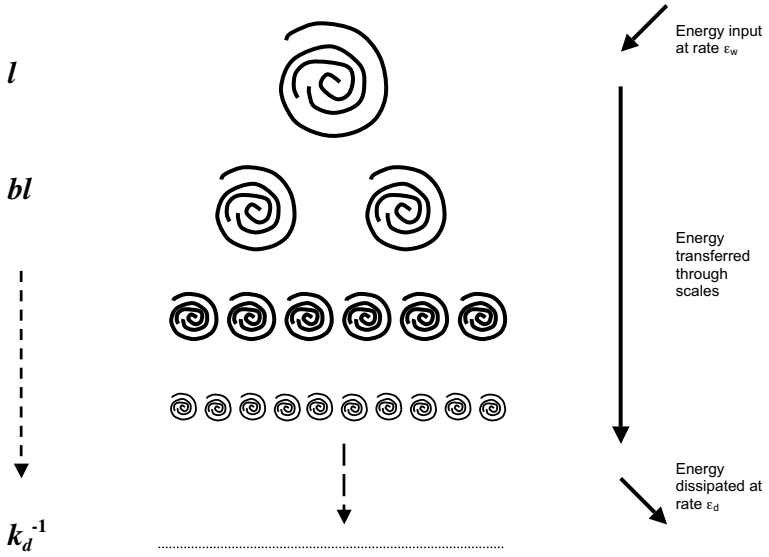

Figure 1.2: Schematic illustrating the Richardson cascade. The kinematics in the form of energy rates are also shown at their respective scales. l and k_d^{-1} are the characteristic length scales for the large and small scales respectively.

acting with each other. So in a simple picture we could view the fluid flow becoming unstable at some critical Reynolds number and developing vortical instabilities due to shear caused by some large velocity differences between neighbouring parts of the flow. These vortical instabilities or eddies would be characterised by some scale l. These eddies would then interact with the flow and with themselves to generate further instabilities in the form of smaller eddies of some characteristic scale bl, where $0 < b < 1$. This process would continue creating more and more complexity and smaller and smaller scales until at some scale, characterised by a wavenumber k_d determined by the viscosity, the motion would be dissipated as heat. This *eddy cascade* picture illustrated in figure 1.2 was first proposed by L. F. Richardson [10] who, influenced by a poem by Johnathan Swift, also put it into rhyme

> Big whorls have little whorls
> Which feed on their velocity,
> And little whorls have lesser whorls
> And so on to viscosity
> (in the molecular sense)

Going back to Fourier space, we can interpret the Richardson cascade as a cascade through the scales represented by the modes k. We expect that two types of interaction can take place between two eddies represented by the wavenumbers k_A and k_B. If one eddy is much larger than the other we expect that the larger eddy will have a tendency to passively convect the smaller eddy rather than shearing it. The latter results in a transfer of energy whilst the former simply changes the phase of the smaller eddy. In the case of two eddies of similar size, however we expect to get an appreciable amount of shear rather than convection. This implies that energy transfer is more likely to occur between two modes of similar size, $k_A \sim k_B$, thus allowing one to assume that energy transfer is a local (in k-space) phenomenon. Thus we can see that the Richardson cascade can also be interpreted to be a local in k-space *energy cascade*. This concept will be important in the next section.

1.4.4 The 1941 Kolmogorov hypotheses

The assumption of the local nature of the turbulence energy transfer seems to imply that we can expect the properties of small scales of well developed turbulence to be independent of the details of the large scales i.e. as the cascade develops we expect the turbulence to have 'forgotten' how it was created. This has important implications of which the most important is that for small scales, or large wavenumbers, the turbulence exhibits *universal* behaviour. The second important implication is that the turbulence is locally isotropic, since any anisotropies would mean that the details of large scales still have an influence. This hypothesis based upon the Richardson cascade was first proposed by A. N. Kolmogorov (1941) [11] in his first *hypothesis of similarity* which states[2] that

> for locally isotropic turbulence, the energy spectrum $E(k)$ is uniquely determined by the quantities ν and ε.

Using dimensional analysis we can then obtain a form for the energy spectrum as

$$E(k) = \nu^{5/4}\varepsilon^{1/4} f(k/k_d), \tag{1.41}$$

[2] In the original paper Kolmogorov referred to the probability distribution (which is directly related via the spectral energy density) and worked with velocity structure functions in x-space rather than the energy spectrum in k-space. We have modified the Kolmogorov hypotheses to take account of these changes and we are also simplifying matters by considering the case of stationary turbulence only, so that the dissipation rate ε is a constant.

where f is an unknown dimensionless function of universal form and

$$k_d = \left(\frac{\varepsilon}{\nu^3}\right)^{1/4} \tag{1.42}$$

is the *Kolmogorov dissipation wavenumber* used to scale k. This is known as the **dissipation** wavenumber because if taken with the associate velocity scale

$$v_d = (\nu\varepsilon)^{1/4}, \tag{1.43}$$

to form a Reynolds number, we get the result

$$Re = \frac{k_d^{-1} v_d}{\nu} = 1, \tag{1.44}$$

which when compared with (1.4), tells us that $k \sim k_d$ is the wavenumber at which dissipation effects start to dominate.

The second hypothesis of similarity states that

> for sufficiently large Reynolds numbers there exists an inertial subrange such that the energy spectrum $E(k)$ becomes independent of the viscosity ν and is uniquely determined by the quantity ε,

so that dimensional analysis determines that the dimensionless function f must take the form

$$f(k/k_d) = \alpha(k/k_d)^{-5/3}, \tag{1.45}$$

where α is a universal constant known as the *Kolmogorov constant*. Substituting (1.45) in (1.41) we obtain the celebrated *Kolmogorov spectrum*

$$E(k) = \alpha\varepsilon^{2/3}k^{-5/3}, \tag{1.46}$$

which can also be considered the form of the entire energy spectrum for the case when the Reynolds number tends to infinity.

1.5 Renormalization

1.5.1 Closure & renormalized perturbation schemes

The term '*renormalization*' has its origins in quantum field theory were it was used to remove divergences in perturbation expansions. For example in Quantum Electro-Dynamics one encounters 'loop' integrals of the form

$$\int_0^\infty \frac{dp}{p}, \tag{1.47}$$

that introduce two sources of divergence; one from below as $p \to 0$ which is known as an *infrared* (IR) divergence; and one from above which is known as an *ultraviolet* (UV) divergence. A theory was said to be 'renormalizable' if these such divergences could be removed. The process in which these were renormalized involved the redefinition of observable quantities by introduction of infinite counter-terms or corrections to the bare parameters of the theory e.g. the mass or charge.

The concept of renormalization, however, need have nothing to do with divergences; and in many cases outside of quantum theory, is not used in this context. In fact, the basic idea of renormalization can be illustrated by the use of mean-field theories [12], in which we have some sort of complex interaction that is modelled by a 'smearing out' of the interaction. This can be shown in the Debye-Hückel theory of the *screened potential* of an electron in an electrolyte, which is one of the earlier examples of a mean-field theory. In this theory, consideration is given to model the potential of an electron which alone would have a coulomb potential at a distance r given by

$$V(r) \simeq \frac{e}{r}, \tag{1.48}$$

(where e is the electron charge) but in this scenario is surrounded by a cloud of other electrons. The potential of our electron amongst this cloud is now changed due to the screening effect of the other electrons surrounding it. In Debye-Hückel theory the new potential is given by

$$V(r) \simeq \frac{e \exp(-r/l_D)}{r}, \tag{1.49}$$

where l_D is the Debye-Hückel length set by the electron number density and the temperature. What is important here for our discussion is that our 'bare' electron charge e has been replaced by a 'renormalized charge' $e\exp(-r/l_D)$.

Turbulence has also had attempts to be renormalized. The earliest account of this can be seen by the introduction of a turbulence diffusivity by Boussinesq in 1877 [13], followed many decades later by the eddy-viscosity of Heisenberg [14] and the Quasi-Normal hypothesis of Proudman and Reid [15] and Tatsumi [16]. In these schemes, the renormalization procedure was introduced by guessing or approximating a relationship between velocity field moments of different order; and thus were known as closure models. The similarities with renormalized quantum field theories came with the seminal work of Kraichnan with the Direct Interaction Approximation (DIA) [17, 18]. This introduction of turbulence research to what became known as Renormalized Perturbation Theories or RPT was given a more comprehensive and universal formulation with the diagrammatic techniques of Wyld [19]. This topic will be discussed later when we review RPT schemes in turbulence theory.

1.5.2 The Renormalization Group: RG

Earlier, we mentioned that the removal of divergences in quantum field theory involved the introduction of infinite counter-terms to cancel the divergent terms. Another way to look at this method was to see it as introducing a cut-off to limit the divergences. However, this introduces a scale into the system which was not there before; the method of canceling one infinity from another was not unique. For a universally valid theory this method should not depend upon any arbitrary choice of this new scale introduced. In the end we demand scale invariance: the renormalization procedure should be independent of the type of scale we associate with it. This introduced the concept of *renormalization invariance* and Stueckelberg and Petermann [20] first observed that transformation groups could be defined which relate different reparametrisations or scales. They introduced these transformation groups as "groupes de normalization" translated as "renormalization groups" or RG. This was extended in a much more thorough way by Gell-Mann and Low [21]. Later Callan and Symanzik [22, 23] systematically derived from the recursion relations of the RG transformations, the set of differential equations that are now generally

used as the RG equations which embody renormalization invariance, known as the Callan-Symanzik equations.

As discussed by Cardy [24], the renormalization group is not a kind of “universal machine through which any problem may be processed”. RG is merely a framework which has to be formulated in a correct way to the nature of the problem being studied. Thus we find that removal of divergences is not the only application of RG. Wilson realised that these methods had a far wider field of application in the scaling theory of critical phenomena that was being formulated by Kadanoff and others in the early seventies. We will leave the discussion of this method to the next chapter, as this is the method which this thesis will be primarily concerned with.

Essentially, the RG transformation is a scaling symmetry transformation and like all other symmetry transformations in physics, quantities and systems that are invariant to such transformations yield deep insights into the physics of these systems. For example, temporal symmetries indicate energy conservation and translational symmetries reveal momentum conservation. RG transformations indicate system scale invariance. The system invariant under a RG transformation is scale-free and reveals the universal nature of the physics of such systems, away from boundaries and external conditions. In this sense the notion of RG transformations is very deep indeed.

An excellent and clearly presented account of the historical development of the Renormalization Group and the concepts therein can be found in the lecture given by Wilson on the presentation of his Nobel Prize in 1982 [25].

1.5.3 Computation and large eddy simulations

In the past two decades the increase of computational power has meant that systems with large degrees of freedom may now be tackled by directly simulating them. The idea of simulating the NSE was first introduced by Orszag in 1969 [26], and since then computation power has increased by some orders of magnitude. These approaches are known as direct numerical simulations or DNS. However, even with such advances we are only just beginning to reach Reynolds numbers which are comparable to flows of real interest. The problem lies with the way the degrees of freedom, represented by the number of points

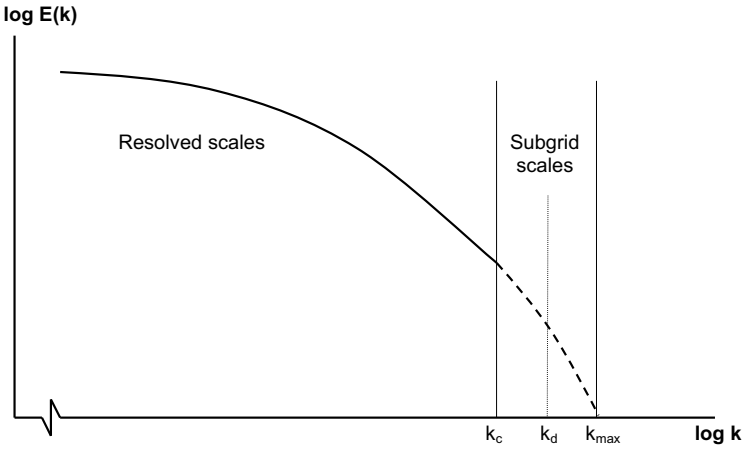

Figure 1.3: A schematic showing, in the energy spectrum picture, the partitioning of scales in large eddy simulations. k_{max} is the UV cut-off, k_d is the Kolmogorov dissipation wavenumber, and k_c is the partitioning wavenumber.

simulated, scales with Reynolds number [27]

$$N \sim Re^{9/4}. \tag{1.50}$$

To give an example, a simulation with 256^3 would yield a sufficient enough resolution to correctly simulate a Reynolds number of $Re \sim 2 \times 10^3$, which is just approaching the transition of laminar to turbulent flow in pipes: a far cry from well developed turbulence such as in the atmosphere or oceans. To date the largest simulations being conducted are around the $\sim 2048^3$ range (some even up to $\sim 4096^3$) for only a few time steps, running on the Earth simulator in Japan [28]

Although advances in computation are rapid, we still have a fair amount of time to wait until we can be firmly in decent Reynolds number territory. Thus there is a need for other methods, albeit approximate, which can help. One method involves the systematic reduction of the degrees of freedom needed to be simulated. This approach is known as a large eddy simulation or LES. Like the name suggests, LES seeks to simulate only the largest or *resolved* modes in a simulation which are of interest in practical flows such as the flow of air around the fuselage of a jet plane. The remaining effects of the smaller or *subgrid* scales on the resolved scales are approximated in some way; the latter approximations being the distinguishing feature of different proposed models. The general idea of an LES is illustrated in figure 1.3. Unlike a normal DNS,

where all modes up to some $k \sim \mathcal{O}(k_d)$ are simulated, LES simulates only modes up to an introduced cut-off $k_c < k_d$. The effects of the subgrid scales $k_c \leq k \leq k_{max}$ are then modelled by augmenting some transport parameter/s; in most cases a wavenumber dependent viscosity in the same spirit as the *eddy viscosity* of Heisenberg [14], is introduced. Although less accurate than a DNS, the success of an LES depends upon how accurately one resolves the large scale modes represented by the choice of cut-off k_c and the grid-points resolution, *and* also upon the way one models the eddy viscosity. These are usually set by correctly modelling the kinetics of the system and in most cases rely on other theories such as some Eulerian closure of the RPT kind. In particular we require that the total energy $E(t)$ and the energy dissipation $\varepsilon(t)$ remain the same i.e. for stationary turbulence (see (1.40)) this would require

$$\int_0^{k_{max}} dk E(k) \simeq \int_0^{k_c} dk E(k) \ , \tag{1.51}$$

and

$$\int_0^{k_{max}} dk 2\nu k^2 E(k) \simeq \int_0^{k_c} dk 2\nu_C k^2 E(k) \ , \tag{1.52}$$

where ν_C is the enhanced eddy viscosity. Generally, these requirements can only be met as an approximation. Note that by writing the conditions in the above way we are assuming that our UV cut-off k_{max} is itself suitably large enough to retain the kinetics of the system. A recent study by McComb, Hunter and Johnston [29] has shown that for a DNS to preserve the kinetics one needs to set the UV cut-off at a minimum of $k_{max} \sim 1.8 k_d$.

A relatively newer approach is to set the LES by way of a RG method, where the cut-off is chosen at the wavenumber at which the RG reaches its fixed point and the eddy viscosity is represented by the fixed point renormalized viscosity. Concepts such as the fixed point will be discussed in more detail in Chapter 2.

1.6 Overview of thesis

The bulk of this thesis will be split into two parts, which detail the assessment of the two renormalization approaches briefly touched upon in section 1.5, and in particular focusing on the renormalization procedures put forward by the proponents of the school of McComb. The first part of this thesis will discuss RG

approaches to turbulence, with a focus on applications to reduce the degrees of freedom in an LES. This will be reviewed in Chapter 2. In Chapters 3–5 we will be discussing the detailed calculations, problems and possible solutions of a particular RG method put forward by McComb *et. al.*; followed in Chapter 6 with one application of this method to modelling the turbulent advection of a passive scalar. The second part of this thesis will begin by reviewing Eulerian RPT closures in Chapter 7. In Chapter 8 we put forward a new formulation of the *local energy transfer theory* (LET) of McComb *et. al.* which resolves some problems of previous derivations. This is followed by introducing a single-time Markovian closure based on LET. We conclude in Chapter 9 with a discussion on the advances in this thesis and the outstanding problems still facing us.

Part I

Renormalization Group Methods

Chapter 2

Dynamical Renormalization Group analysis and turbulence

In the introduction we briefly described the notion of renormalization. In particular we discussed the renormalization group (RG) as being a special form of renormalization that involves a description of a system in terms of scales. The power of RG is then manifested in the identification of scale-free behaviour in the system being studied and in the 'universal' aspects which can be identified as a result. In this chapter we will be reviewing RG by looking at the application of this powerful tool in the statistical study of turbulence. After a general outline of the method, we will briefly discuss large-eddy simulations (LES) as our primary motivation for studying RG (in the context of this thesis) i.e. the ability of RG to reduce the degrees of freedom in a system so as to facilitate computation. As turbulence is a non-equilibrium phenomena, we will be looking at RG applied to dynamical systems, as first studied by Ma and Mazenko [30] and others [31, 32]. In particular this will involve a discussion of the seminal work of Forster, Nelson and Stephen (FNS) [33, 34], which was the first application of RG to the Navier-Stokes equation. This will be followed by a broad review of some of the main attempts at formulating RG as applied to turbulence, and of the problems encountered therein.

2.1 The RG procedure

RG is primarily a transformation concerned with determining scale invariance in a system. By scale invariance we mean that a system characterised by a particular scale should behave in the same way as if we had characterised it by any other different scale. The detailed strategy of checking for scale invariance, and thus the RG transformation, differs for each system studied but what follows is a general method where the incompressible NSE as given by equation (1.10), is used as an example dynamical system. An excellent, brief and simple review of RG and scaling can be found in Stanley [35], and in the example which appears in Chowdhury and Stauffer (2000, 486-488) [36]; where illustrations are primarily given in terms of percolation phenomena.

We proceed by introducing a UV cut-off, $k_{max} \equiv k_0$, as is normal in these studies; and it is chosen such that it retains most of the physics (see previous chapter). For the purposes of this calculation, one could say that k_0 is the relevant length scale characterising this system. We write the NSE in shorthand notation so as to strip away any distractions from the quintessential RG procedure

$$\left[\frac{\partial}{\partial t} + \nu^0 k^2\right] u_k = \lambda_0 M_k u_j u_{k-j} + f_k^0 \ , \tag{2.1}$$

where the equation is defined on the wavenumber interval $[0, k_0]$, and where only the wavenumber has been retained as a subscript and the time and tensor indices as well as the convolutions are implicitly assumed; λ_0 is a book-keeping parameter (which is eventually set to unity), and the 0 superscripts, as will later become clear, are keeping track of the renormalization cycle.

Now there is no reason why it should not be theoretically possible for one to selectively assess the effects of the small scale velocity fluctuation modes on the dynamics of the remaining ones. Thus, we proceed by trying to project the equations of motion onto the k-space spanned by the wavenumbers representing large scales i.e. smaller wavenumbers. This we do by partitioning the effects of the velocity (fluctuations) field into those acting at high wavenumbers (small scales) u^+, and those acting at low wavenumbers (large scales) u^-, such that

$$\begin{aligned} u_k &= u_k^- + u_k^+ \\ &= \theta(k_1 - k)u_k + \theta(k - k_1)u_k \ , \end{aligned} \tag{2.2}$$

where $k_1 < k_0$ is the wavenumber partitioning large and small scales, and the Heaviside unit-step function $\theta(k)$ is given by

$$\theta(k) = \begin{cases} 1 \; \forall k \geq 0 \\ 0 \; \forall k < 0 \end{cases} . \tag{2.3}$$

The partitioning wavenumber k_1 is given by $k_1 = hk_0$ where $0 < h < 1$.

Multiplying equation (2.1) through by $\theta(k_1 - k)$ and substituting (2.2), we obtain the equation for the low-k modes

$$\left[\frac{\partial}{\partial t} + \nu^0 k^2\right] u_k^- = \lambda_0 M_k^- \left[u_j^- u_{k-j}^- + 2u_j^- u_{k-j}^+ + u_j^+ u_{k-j}^+\right] + f_k^{0-} . \tag{2.4}$$

Similarly, multiplying (2.1) by $\theta(k - k_1)$ gives us the equation for the high-k modes

$$\left[\frac{\partial}{\partial t} + \nu^0 k^2\right] u_k^+ = \lambda_0 M_k^+ \left[u_j^- u_{k-j}^- + 2u_j^- u_{k-j}^+ + u_j^+ u_{k-j}^+\right] + f_k^{0+} . \tag{2.5}$$

Theoretically, we can solve[1] equation (2.5) for the high-k modes

$$u_k^+ = g[u_k^-, k] , \tag{2.6}$$

and thus eliminate the u^+ dependence in equation (2.4). This gives the dynamical equation for the low wavenumber modes that we originally sought

$$\left[\frac{\partial}{\partial t} + \nu^0 k^2\right] u_k^- = \lambda_0 M_k^- u_j^- u_{k-j}^- + f_k^{0-} + \Omega[u_k^-, k] , \tag{2.7}$$

where

$$\Omega[u_k^-, k] = \lambda_0 M_k^- \left\{2u_j^- g[u_{k-j}^-, k-j] + g[u_j^-, j] g[u_{k-j}^-, k-j]\right\} , \tag{2.8}$$

has been introduced to save unnecessary clutter. Equation (2.7) represents the *coarse-grained* description of the system in terms of the low wavenumber modes only.

We are looking for scale invariance in the system. For this, we require that the dynamical equations look the same. This requirement is known as form invariance. Comparing our original NSE (2.1) defined on $[0, k_0]$ and our new

[1] This inevitably involves some sort of perturbation scheme.

low wavenumber NSE (2.7) defined on $[0, k_1]$, we see that the equations differ by the addition of the functional $\Omega[u_k^-, k]$. To do something about this extra term, we attempt to write it in terms of the other terms in the equation. This can normally only be done as an approximation as the coarse-graining will in general introduce more complex new couplings in addition to the original NSE non-linearities; and inevitably this will involve some sort of averaging. This process of requiring form invariance will normally redefine one or more of the transport parameters (λ_0, ν^0, f_k^0) such that our new equation will now be

$$\left[\frac{\partial}{\partial t} + \widetilde{\nu}^0 k^2\right] u_k^- = \tilde{\lambda}_0 M_k^- u_j^- u_{k-j}^- + \widetilde{f}_k^{0-} \ , \tag{2.9}$$

where the tilde denotes the new augmented parameters.

Now the equations have the same form, we must realise that the comparison we made earlier was not fair: the equations (2.9) and (2.1) which are being compared are defined on different intervals. Thus we rescale equation (2.9) so that it is defined over the interval $[0, k_0]$ like our original NSE. This will, again, in general change the viscosity, book-keeping and force parameters such that our new dynamical equation becomes

$$\left[\frac{\partial}{\partial t} + \nu^1 k^2\right] u_k' = \lambda_1 M_k u_j' u_{k-j}' + f_k^1 \ , \tag{2.10}$$

where the prime on the velocity field denotes that it only comprises the low wavenumber modes and the higher modes or degrees of freedom have been renormalized into the transport parameters of the system. This completes the renormalization procedure. Does this now imply that we have scale invariance? Unfortunately, no. The transport parameters have been changed, and thus so has the system. Only when these parameters have also become invariant to the renormalization procedure, do we have scale invariance. Thus we repeat the renormalization procedure, now with the system parameterised on the scale $k_1 = hk_0$, and check for scale invariance in the next coarse-grained picture which will be parameterised on the scale $k_2 = hk_1 = h^2 k_0$. This set of renormalization transformations from one coarse-grained picture to the next constitutes what is known as the renormalization group. Only when all the system parameters stop changing i.e have reached a *fixed point* under this renormalization group transformation, we attain scale invariance in the system behaviour. The parameter h can now be identified as a measure of the frac-

tion of the degrees of freedom which are eliminated at each stage of the RG iteration.

Both the first stage of coarse-graining and the second stage of rescaling constitute the renormalization procedure. Comparing with renormalized perturbation theories in turbulence (see later in Chapter 7) which can have uncontrolled approximations, RG can be seen as several little renormalization steps, which have the two primary benefits of looking for scale invariance and being able to control approximations by only renormalizing in an area where the approximations are valid. Furthermore, in the context of controlling approximations, since most coarse-graining schemes involve a perturbation expansion in the book-keeping parameter λ, it is desired that our approximation stay within an area where any renormalization of λ will keep its value small (less than unity) such that a lower order truncation is justified.

However, RG also has its own disadvantages. These are normally linked with our own ignorance rather than any deficiencies with the RG procedure. We need to know the existence of a fixed point *a priori*. In the introduction we have already discussed the existence of one such fixed point in turbulence, which corresponds to the Kolmogorov scale-free inertial range. On top of this we have to construct a suitable coarse-graining scheme. This latter hurdle is non-trivial and is usually what distinguishes one RG scheme from another.

We will finish this section with a summary of the RG shell by shell renormalization scheme:

1. Project the equations of motion onto the k-space spanned by modes u^- with $0 \leq k \leq hk_0$, and coarse-grain system by eliminating the modes u^+ corresponding to wavenumbers $hk_0 \leq k \leq k_0$. This introduces new interaction terms. Demanding form invariance will redefine the transport parameters which enter the new reduced equations of motion.

2. Rescale all relevant variables such that the new equation is defined over the same interval as the older equation.

3. Repeat until the renormalized parameters do not change i.e. have reached a fixed point.

2.1.1 Cross and Reynolds interaction terms

Before we move onto reviewing the attempts to apply RG to turbulence, we will briefly say something about the main hurdles which are facing anyone who seeks to go down this road. Looking at the low-k NSE as given by equation (2.4), we see that the first interaction term denoted by u^-u^- will not need to be modified by the coarse-graining. However, the other two terms which we will call the *cross term* u^-u^+ and the *Reynolds term* u^+u^+, do have to be coarse-grained as they involve the u^+ modes. It is the coarse-graining of these terms which presents the major obstacle in forming a suitable RG scheme for the NSE.

2.1.2 Spectral LES applications

In this thesis we aim not to do the full conventional study as in the calculation of critical exponents etc. that RG is normally associated with. Instead we utilise the nature of RG to reduce the degrees of freedom in a system so as to use it to obtain a consistent subgrid model for an LES. Recall from Chapter 1 that the two major components of an LES is a choice of a **cut-off** between resolved and subgrid modes and an associate **subgrid model** (usually an enhanced viscosity) to account for the missing modes. From the above summary, we can see that RG supplies us with both of these components. The cut-off is represented by the wavenumber at which we stop the RG iteration i.e. at the approximate (within a percentage error) wavenumber at which we attain the fixed point in the renormalized parameters; and the subgrid model is obtained from the enhanced fixed point renormalized viscosity. To obtain these two components we need to remove the effects of the rescaling from the renormalized viscosity, and obtain the cut-off wavenumber from $k_c = h^N k_0$, where N is the number of steps taken to reach the fixed point. This assumes that only the viscosity is enhanced by the coarse-graining, and that the force and book-keeping (coupling) parameters are only renormalized from the rescaling part of the RG algorithm.

An important point to note is that ideally we would want the subgrid model to retain all the dynamics and kinetics of the missing modes. The dynamics in particular require all the phase information to be retained. In practice this does not occur as the retention of *all* the phase information is just another manifestation of the formidable turbulence problem, and would require a detailed

realisation dependent subgrid model. Thus, all LES models to date resort to relying on the correct retaining of the kinetics. This implies that the total energy $E(t)$ and dissipation rate $\varepsilon(t)$ be correctly retained in any LES subgrid model (see section 1.5.3). Thus in the context of RG we require that at each stage of the RG iteration, both the total energy of the system and the dissipation rate be invariant to the RG transformation; such that in the case of statistically stationary turbulence where $E(t) \equiv E$ and $\varepsilon(t) = \varepsilon$ (see (1.40))

$$\int_0^{k_0} E(k)dk = \int_0^{k_1} E(k)dk \ , \tag{2.11}$$

and

$$\int_0^{k_0} 2\nu^0 k^2 E(k)dk = \int_0^{k_1} 2\nu^1(k) k^2 E(k)dk \ . \tag{2.12}$$

Of course, depending on how faithful our coarse-graining approximation is to the subgrid kinetics, both these criteria can only be met as an approximation. As mentioned earlier in Chapter 1, it has been shown by McComb, Hunter & Johnston [29], that even when the upper cut-off is quite small, the effect on any reduction of the total energy E is insignificant. However, the energy dissipation ε_d needs to be correctly accounted for in the subgrid model to ensure (2.12) is met. The faithfulness of the RG coarse-graining model presented in this thesis will be discussed in more detail in Chapter 5.

A summary of the RG procedure and its application to LES, as represented by the kinetic energy spectrum picture, is illustrated in figure 2.1.

2.2 RG based on an ϵ-expansion

The following review is based upon subdividing the RG schemes into two classes. The first is one in which a perturbation expansion in terms of a small parameter ϵ is used, where ϵ is a combination of dimension and forcing exponents; and the prescribed statistics of a random stirring force is used to average over infinitesimal shells of degrees of freedom. A recursion relation in the form of a differential equation is then solved to obtain fixed points. The second class is one in which finite shells of degrees of freedom are removed, which result in a recursion relation that is iterated upon to obtain fixed points. The distinction between these two classes will become more apparent as we proceed

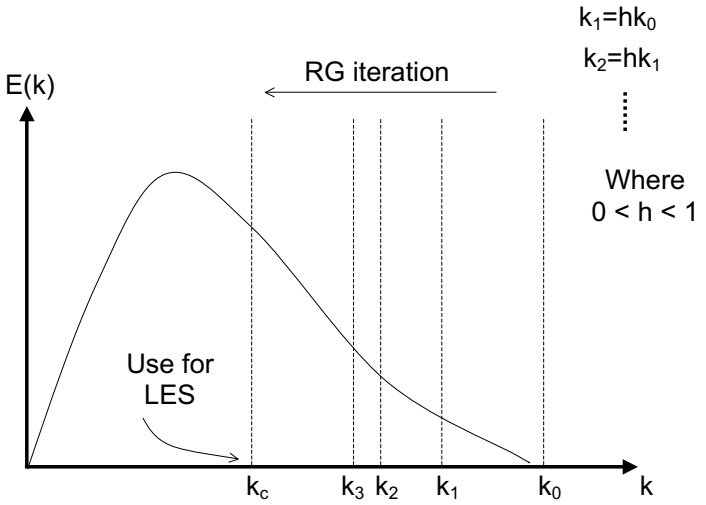

Figure 2.1: Evolution of the RG iteration with the rescaling removed. Shells of degrees of freedom are removed iteratively which renormalize the transport parameters of the system. k_c corresponds to the approximate wavenumber at which the fixed point in the transport parameters is attained (up to an accuracy set by us). This is the wavenumber which is then used for an LES cut-off, with its associate subgrid model that is obtained from the renormalized/effective viscosity.

with the review. One should note that there exists another use of RG in the study of turbulence, where field theoretic renormalization techniques based on a generating functional introduced by Martin, Siggia and Rose [37], are used to calculate anomalous scaling and intermittency effects [38, 39, 40, 41, 42]. Although interesting in their own right, their validity is limited to artificial cases due to various assumptions of Gaussian velocity fields, $k \to 0$ limits etc. which do not apply to the case of turbulence. Moreover, much of the machinery of quantum field theoretic RG is taken over to apply to the NSE, without a sufficient precautionary study into the applicability of such techniques to classical dynamical systems. Thus, due to these reasons and to the lack of apparent connection to the aims of the RG scheme in the present work, discussion of these techniques will be omitted from this review.

2.2.1 Forster, Nelson and Stephen (FNS) (1977)

As mentioned earlier, the pioneering work of FNS [33, 34] in the application of dynamic RG to the NSE, was based upon the earlier work of Ma and Mazenko

[30]. However, although FNS studied the NSE system, their study did not actually describe turbulence due to their UV cut-off k_0 being taken to be below the inertial range and their requirement of the theory being valid in the infra-red $k \to 0$ limit. Thus the cascade effects which are the quintessential signatures of turbulent phenomena were omitted and by the title of the FNS seminal paper, their study was restricted to the theory of large-distance IR behaviour of a randomly stirred fluid which is also known to be asymptotically scale free and thus exhibits a fixed point in the limit $k \to 0$.

The FNS RG scheme begins with equations similar to (2.4) and (2.5); however FNS worked with temporal as well as spatial Fourier transformed equations, so their equations for the high and low wavenumber velocity field take the form

$$u_k^- = \lambda_0 M_k^- G_k^0 \left[u_j^- u_{k-j}^- + 2u_j^- u_{k-j}^+ + u_j^+ u_{k-j}^+\right] + G_k^0 f_k^{0-} \ . \tag{2.13}$$

$$u_k^+ = \lambda_0 M_k^+ G_k^0 \left[u_j^- u_{k-j}^- + 2u_j^- u_{k-j}^+ + u_j^+ u_{k-j}^+\right] + G_k^0 f_k^{0+} \ . \tag{2.14}$$

where the unrenormalized *bare propagator* is given by

$$G_k^0 = \left(i\omega + \nu^0 k^2\right)^{-1} \ , \tag{2.15}$$

ω is the temporal frequency (Fourier conjugate to the time t), and the low-k velocity field is defined on the interval $[0, k_1]$ where $k_1 = e^{-l}k_0$ and l is a measure of the spatial rescaling factor (e^{-l} is equivalent to our earlier h); and the high-k velocity field is defined on $[k_1, k_0]$. FNS take their UV cut-off to be $k_0 \ll k_d$, where k_d is the Kolmogorov dissipation wavenumber (which is a rough indication of where viscosity dominates the fluid dynamics). The force is introduced in the NSE to obtain a statistically steady state, and is assumed to have Gaussian white noise statistics with the specified two-point two-time auto-correlation given by

$$\langle f_\alpha(\mathbf{k}, \omega) f_\beta(\mathbf{k}', \omega')\rangle = 2D(k)(2\pi)^{d+1} P_{\alpha\beta}(\mathbf{k})\delta(\mathbf{k} + \mathbf{k}')\delta(\omega + \omega') \ , \tag{2.16}$$

where the projector $P_{\alpha\beta}(\mathbf{k})$ is given by $\delta_{\alpha\beta} - k_\alpha k_\beta / k^2$, and $D(k)$ is the forcing function. FNS study three models for $D(k)$ which they claim are representative of the two universality classes that describe the infra-red wavenumber properties of the NSE. We will focus on the FNS model B as it exhibits the most

interesting results as well as being the broad model which seems to have been adopted by the many others who came after FNS. The FNS model B corresponds to the forcing function taking the constant form $D(k) = D^0$, however for comparison with later RG schemes we will write the forcing function in a general way as

$$D(k) = D^0 k^{-y} \ , \tag{2.17}$$

where y is the forcing exponent mentioned earlier, and where FNS model B is when $y = 0$.

To proceed with the coarse-graining, we eliminate u^+ by use of a perturbation expansion about the NSE with the nonlinearity switched off i.e. $\lambda^0 = 0$. The perturbation series is formally written as

$$u_k^+ = u_k^{+(0)} + \lambda_0 u_k^{+(1)} + \lambda_0^2 u_k^{+(2)} + \dots \ . \tag{2.18}$$

Substituting this expansion on both sides of equation (2.14) allows us to identify the coefficients of the above expansion by equating similar orders of the book keeping parameter λ_0

$$\begin{aligned} u_k^{+(0)} &= G_k^0 f_k^{0+} \ , \\ u_k^{+(1)} &= G_k^0 M_k^+ \left[u_j^- u_{k-j}^- + 2u_j^- u_{k-j}^{+(0)} + u_j^{+(0)} u_{k-j}^{+(0)} \right] \ , \\ u_k^{+(2)} &= G_k^0 M_k^+ \left[2u_j^- u_{k-j}^{+(1)} + 2u_j^{+(0)} u_{k-j}^{+(1)} \right] \ , \\ & \quad etc. \end{aligned} \tag{2.19}$$

Substituting these coefficient back into the RHS of equation (2.13) and expanding out the $u^{+(0)}$ and $u^{+(1)}$ coefficients in terms of the forcing f^{0+} and bare propagator G^0, gives to second order in λ_0

$$u_k^- = \lambda_0 M_k^- G_k^0 u_j^- u_{k-j}^- + G_k^0 f_k^{0-} + \lambda_0 A + \lambda_0^2 B + \mathcal{O}\left(\lambda_0^3\right) \ , \tag{2.20}$$

where

$$A = 2M_k^- G_k^0 G_{k-j}^0 f_{k-j}^{0+} u_j^- \ , \tag{2.21}$$

and

$$\begin{aligned} B &= 2M_k^- M_{k-j}^+ G_k^0 G_{k-j}^0 u_j^- u_p^- u_{k-j-p}^- \\ &\quad +4M_k^- M_{k-j}^+ G_k^0 G_{k-j}^0 G_{k-j-p}^0 f_{k-j-p}^{0+} u_j^- u_p^- \\ &\quad +2M_k^- M_{k-j}^+ G_k^0 G_j^0 G_{k-j}^0 f_j^{0+} u_p^- u_{k-j-p}^- \\ &\quad +2M_k^- M_{k-j}^+ G_k^0 G_{k-j}^0 G_p^0 G_{k-j-p}^0 f_p^{0+} f_{k-j-p}^{0+} u_j^- \\ &\quad +4M_k^- M_{k-j}^+ G_k^0 G_j^0 G_{k-j}^0 G_{k-j-p}^0 f_j^{0+} f_{k-j-p}^{0+} u_p^- \\ &\quad +2M_k^- M_{k-j}^+ G_k^0 G_j^0 G_{k-j}^0 G_p^0 G_{k-j-p}^0 f_j^{0+} f_p^{0+} f_{k-j-p}^{0+} \,. \end{aligned} \tag{2.22}$$

The next step is to average out the effect of the high-k wavenumbers. This is done using a filtered ensemble average which has the properties that

$$\begin{aligned} \left\langle f^- \right\rangle_{high} &= f^- \,, \\ \left\langle u^- \right\rangle_{high} &= u^- \,, \end{aligned} \tag{2.23}$$

where the subscript $high$ indicates that only high wavenumber terms are averaged. The filtered average for example will give

$$\begin{aligned} \left\langle u^- f^- u^+ f^+ \right\rangle_{high} &= u^- f^- \left\langle u^+ f^+ \right\rangle_{high} \\ &= u^- f^- \left\langle u^+ f^+ \right\rangle \end{aligned} \tag{2.24}$$

i.e. it averages out terms associated with high wavenumbers in the normal ensemble averaged way, and leaves the low wavenumber terms untouched. We now apply the filtered ensemble average on equation (2.20) and evaluate the averages using the statistics of the forcing and noting that

- G^0 is statistically sharp so that it is not affected by any averaging;
- the stirring forces are statistically homogeneous;
- for consistency, the stirring forces have zero mean $\langle f^+ \rangle = 0$ (also arises from next property);
- and the stirring force statistics are Gaussian so that odd order moments vanish.

When this is completed we will get the following expression (see McComb [5]

for details)

$$\begin{aligned} u_k^- &= \lambda_0 M_k^- G_k^0 u_j^- u_{k-j}^- + G_k^0 f_k^{0-} \\ &\quad +\lambda_0^2 2 M_k^- M_{k-j}^+ G_k^0 G_{k-j}^0 u_j^- u_p^- u_{k-j-p}^- \\ &\quad +\lambda_0^2 8 M_k^- M_{k-j}^+ G_k^0 \left(G_j^0\right)^2 G_{k-j}^0 P_j (2\pi)^{d+1} D^0 j^{-y} u_p^- \\ &\quad +\mathcal{O}\left(\lambda_0^3\right) \ , \end{aligned} \tag{2.25}$$

where D^0 is from equation (2.17), d is the space dimension and P_j is just a shorthand for the normal $P_{\alpha\beta}(\mathbf{j})$ projector. Identifying that the term in the third line is linear in u^-, and thus can be written as an increment to the viscosity, we can evaluate the convolution integrals and with some further algebra can write equation (2.25) as

$$\begin{aligned} \left(i\omega + \widetilde{\nu}^0 k^2\right) u_k^- &= \lambda_0 M_k^- u_j^- u_{k-j}^- + f_k^{0-} \\ &\quad +\lambda_0^2 2 M_k^- M_{k-j}^+ G_{k-j}^0 u_j^- u_p^- u_{k-j-p}^- \ , \end{aligned} \tag{2.26}$$

where we have inverted the G_k^0 bare propagator so as to show that the viscosity has been enhanced. To deal with the last term which involves a cubic non-linearity, FNS point out that in the limit $k \to 0$ this term involves an *irrelevant* scaling field. This means that as the RG calculation is iterated, the contribution from this term will tend to zero (see McComb [5] p.356 for more on this calculation). Accordingly we now drop this term and thus attain form invariance of our coarse-grained equations with the augmented viscosity given by

$$\widetilde{\nu}^0 = \nu^0 + \delta\nu^0(0) \ , \tag{2.27}$$

where the argument '0' in the viscosity increment indicates that this was calculated in the limit $k \to 0$; the viscosity increment is given by

$$\delta\nu^0(0) = \frac{A(d) S_d \lambda_0^2 D^0}{(2\pi)^d \left(\nu^0\right)^2 k_0^\epsilon} \frac{e^{\epsilon l} - 1}{\epsilon} \ , \tag{2.28}$$

where the parameters

$$\epsilon = 4 + y - d \ , \tag{2.29}$$

$$A(d) = \frac{d^2 - d - \epsilon}{2d(d+2)} \ , \tag{2.30}$$

and S_d is the area of a unit sphere in d dimensions. As mentioned earlier, although we have used a perturbation expansion in the book-keeping parameter λ_0, in practice what is actually used is a modified strength parameter given by

$$\bar{\lambda}_0(k_0) = \lambda_0 \sqrt{\frac{D^0}{(\nu^0)^3} k_0^{-\epsilon}} \ , \tag{2.31}$$

the origins of which can be seen if we had worked in modified units where time is in units of $1/\nu^0$, velocity in units of $\sqrt{D^0/\nu^0}$, and force in units of $1/\sqrt{D^0\nu^0}$; i.e. if we had made the NSE dimensionless, the effect of this would have been a modified 'vertex' (book-keeping parameter) $\bar{\lambda}_0$, which can be seen to be a local (in wavenumber) Reynolds number based on the wavenumber k_0.

The next step involves re-scaling all the basic variables. Doing this introduces a set of homogeneous scaling relations, which are used to fix the various scaling exponents e.g. we scale the spatial and temporal frequencies like

$$k' = ke^l \ , \quad \omega' = \omega e^{\alpha(l)} \ , \tag{2.32}$$

where $\alpha(l)$ is to be determined. Since l (the scaling parameter) is infinitesimal for an infinitesimal shell removal, FNS write their recursion relations for the renormalized transport parameters in differential equation form

$$\frac{d\nu(l)}{dl} = \nu(l) \left\{ z(l) - 2 + \frac{A(d) S_d}{(2\pi)^d} \left(\bar{\lambda}(l)\right)^2 \right\} \ , \tag{2.33}$$

$$\frac{dD(l)}{dl} = 0 \tag{2.34}$$

$$\frac{d\lambda(l)}{dl} = \left[-1 - \frac{1}{2}d + \frac{3}{2}z(l) \right] \lambda(l) \ , \tag{2.35}$$

where

$$z(l) = \frac{d\alpha(l)}{dl} \ , \tag{2.36}$$

and for the modified coupling we have

$$\frac{d\bar{\lambda}(l)}{dl} = \frac{\bar{\lambda}(l)}{2} \left[\epsilon - 3 \frac{A(d) S_d}{(2\pi)^d} \bar{\lambda}^2(l) \right] \ , \tag{2.37}$$

where more accurate values of the renormalized parameters are then obtained by integrating these differential equations. Fixed point values are then found when the renormalized parameters become independent of l. For $\epsilon \leq 0$ we find

that $\bar{\lambda}(l)$ can be made as small as we want depending on the choice of l, and tends to the trivial fixed point $\bar{\lambda} = 0$ as $l \to \infty$. However, for $\epsilon > 0$ we obtain a non-trivial fixed point, which is given by setting the RHS of equation (2.37) to zero. This gives the value of the fixed point at

$$\bar{\lambda}^* = \left(\frac{\epsilon (2\pi)^d}{3A(d)S_d} \right)^{1/2} . \tag{2.38}$$

Substitution of this result into equation (2.33) yields for the fixed point l independent viscosity as requiring that

$$z = 2 - \frac{\epsilon}{3} , \tag{2.39}$$

which then fixes the temporal frequency (or dynamic) scaling factor $\alpha(l)$ in equation (2.32). From equation (2.38) for the renormalized coupling parameter, we can now see why this approach is known as the ϵ-expansion, as we are effectively truncating a perturbation series in terms of the reduced coupling and hence ϵ.

From a homogeneity relationship for the velocity correlation, FNS then find the restrictions for the scaling factors (for when $d < 4$) for the energy spectrum which is given by

$$E(k) \sim k^{-5/3+2(d-y)/3} , \tag{2.40}$$

and for which if we take $d = 3$ and $y = 0$ (FNS model B), gives $E(k) \sim k^{1/3}$ valid in the limit $k \to 0$.

Although we could work with varying the spatial dimension d for a fixed forcing exponent y as FNS originally did, in practice and in most later works, researchers tend to work with the forcing exponent y for a fixed $d = 3$ instead. Thus contrary to the use of the ϵ-expansion in the study of critical phenomena, in turbulence problems the expansion parameter ϵ is the deviation of the power exponent y of the random forcing from the critical value rather than the deviation of the space dimensionality from the upper critical dimension, although FNS did originally use it in this way.

This description of the FNS RG scheme, although quite detailed, sets the relevant background for the next schemes that we will be discussing. It will also serve as a good judge of the degree of rigour and shed a light on some pitfalls and wrong uses of RG that can be found in later schemes which seek to extend

the FNS RG analysis to the case of actual fully developed turbulence.

2.2.2 Fournier and Frisch (1983)

The work of Fournier and Frisch [43] started by looking at critical spatial dimensions where the energy cascade in a turbulent system changes direction from being in the UV direction for normal $d = 3$, and reversing to the IR direction for $d \lesssim 2.05$. Their next paper however attempted to do a calculation on the same lines as FNS, with a white noise statistics force auto-correlation given by equation (2.16) and the forcing function of the form

$$D(k) = \frac{D^0}{4\pi} k^{1-\epsilon} \ , \tag{2.41}$$

where the forcing function can be related to the energy injection rate spectrum $W(k)$ by

$$W(k) = 2D^0 k^{3-\epsilon} \ . \tag{2.42}$$

Fournier and Frisch, however, had in mind the eddy viscosity of the LES form and were particularly interested in calculating values of universal constants: primarily the Kolmogorov constant prefactoring the inertial range $k^{-5/3}$ energy spectrum. They did this by building on the work of DeDominicis and Martin [39] who amongst other things remarked that if we choose in the FNS RG scheme, the forcing exponent $y = d$ then equation (2.40) would result in the energy spectrum taking the Kolmogorov form[2]. In a similar way to FNS, they used a perturbation expansion in the local Reynolds number (reduced coupling constant) given by equation (2.31), and derived the differential recursion relation for the renormalized viscosity to be

$$\frac{d\nu(k_c)}{dk_c} = -\frac{D^0}{10\pi^2\nu^2(k_c)} k_c^{-(1+\epsilon)} \ , \tag{2.43}$$

where the k_c is the partitioning cut-off between the high and low wavenumber modes. Solving equation (2.43) gives a form for the renormalized viscosity for a specific wavenumber cut-off k_c

$$\nu(k_c) \simeq \left(\bar{\lambda}(k_c)\right)^{-2/3} \left(D^0\right)^{1/3} k_c^{-\epsilon/3} \ , \tag{2.44}$$

[2]This would however have repercussions with the irrelevant variables neglected in the FNS study (see McComb [5]).

where $\bar{\lambda}(k_c)$ is the renormalized Reynolds number given by

$$\bar{\lambda}(k_c) \simeq \left(\frac{10\pi^2\epsilon}{3}\right)^{1/2} . \tag{2.45}$$

Fournier and Frisch then used the smallness of $\bar{\lambda}(k_c)$ to write the renormalized form for the stationary energy balance equation (see equation (1.30) in Chapter 1) in the limit $\epsilon \to 0$ as

$$2\nu(k_c)k^2E(k) = W(k) , \tag{2.46}$$

which with equations (2.42) and (2.44) can be used to obtain an approximation to the energy spectrum[3] to leading order in ϵ

$$E(k) \simeq \left(\frac{10\pi^2 \left(D^0\right)^2 \epsilon}{3}\right)^{1/3} k^{1-2\epsilon/3} . \tag{2.47}$$

The requirement that $\epsilon \to 0$ immediately stops one from setting $\epsilon = 4$ and recovering the Kolmogorov spectrum. However, their calculation of the prefactor in front of the power law in equation (2.47), set the scene for the work of Yakhot and Orszag (see below). If one can have something of the form of (2.47) with no qualms with the setting of $\epsilon = 4$ then one can theoretically calculate the value of any universal prefactors provided that one can relate the constant D^0 in the forcing spectra to the energy dissipation rate, as in the form of the inertial range Kolmogorov spectrum $E(k) = \alpha\varepsilon^{2/3}k^{-5/3}$.

Now an important comment that needs to be made on the RG method of Fournier & Frisch is that their calculations did not involve any rescaling which is arguably the most important feature of any RG calculation. They claim that their RG method does not require any rescaling. Their work seems only to concentrate on obtaining a renormalized/enhanced viscosity by which they could carry out their stated aims of calculating a universal prefactor. However, without any rescaling, the links between their method and RG seem quite untenable.

[3] One should note here that the calculation in the paper, although not mentioning it explicitly, makes the viscosity dependent upon any wavenumber k instead of just the cut-off wavenumber i.e. $\nu(k_c) \to \nu(k)$. This can be justified by noting that since we are working in the IR asymptotic limit $k \to 0$, it is ok to make this replacement of arguments since $\nu(k_c) = \nu(k)$ is true in the IR asymptotic region.

2.2.3 Yakhot and Orszag (1986)

As mentioned earlier, the work of Yakhot & Orszag [44, 45, 46], built upon the earlier works of FNS and particularly Fournier & Frisch. Now one might question whether any work that builds upon the work of Fournier and Frisch and retains their neglect of the rescaling, is appropriate to discuss in the context of RG. However it is instructive to do so, as the work of Yakhot and Orzsag set about a flurry of activity in the period between the mid 80's and early 90's (and arguably to this day).

Fortunately, much of the background behind the work of Yakhot & Orszag has already been covered above, so we need only discuss results and problems. The forcing auto-correlation is the same as Fournier & Frisch, and the RG differential recursion relation for the renormalized viscosity is essentially similar also

$$\frac{d\nu(k_c)}{dk_c} = -\frac{A(d)S_d D^0}{(2\pi)^d \nu^2(k_c)} k_c^{-(1+\epsilon)} \,, \tag{2.48}$$

where $A(d)$ is given by equation (2.30). Solving this for the fixed point gives the renormalized viscosity

$$\nu^*(k_c) \simeq \left(\bar{\lambda}^*(k_c)\right)^{-2/3} \left(D^0\right)^{1/3} k_c^{-\epsilon/3} \,, \tag{2.49}$$

where the renormalized fixed point reduced coupling is given by

$$\bar{\lambda}^*(k_c) = \left(\frac{3}{\epsilon} A(d)\right)^{-1/2} \,. \tag{2.50}$$

Yakhot & Orszag then argue that since the renormalized viscosity is a function of the wavenumber k_c (the cut-off) which is essentially arbitrary, we can therefore replace it with the continuous wavenumber k and thus get a wavenumber dependent viscosity which by definition holds over the entire k-space region we are working in

$$\nu^*(k) \simeq \left(\frac{3}{\epsilon} A(d) D^0\right)^{1/3} k^{-\epsilon/3} \,. \tag{2.51}$$

This is a bold assumption by itself since the renormalized viscosity was calculated such that it is valid only in the limit $k \to 0$.

The next step involves a similar approximation to Fournier & Frisch in obtaining

the energy spectrum to leading order in ϵ

$$E(k) \simeq \left(\frac{S_d^2}{A(d)(2\pi)^{2d}} \frac{\left(D^0\right)^2 \epsilon}{3} \right)^{1/3} k^{1-2\epsilon/3} \, . \tag{2.52}$$

Yakhot & Orszag now make the most bold of assumptions. They extend their cut-off such that $k_c \sim k_d$ where k_d is the Kolmogorov dissipation wavenumber. This implies that their theory now encompasses all of the inertial range and beyond, and thus is being applied to actual turbulence. Accordingly they set $\epsilon = 4$ to recover the Kolmogorov spectrum in equation (2.52) . From equation (2.29) we see that (for $d = 3$) this amounts to setting the forcing exponent $y = d = 3$. This now opens up the proverbial 'can of worms' in two ways:

- The most disastrous result of setting $\epsilon = 4$ is that the ϵ expansion relies on the fact that ϵ is small so that a leading order truncation is justified, both in the coarse-grained dynamical equation and also in the derivation of equation (2.52) for the energy spectrum. ϵ being large introduces higher order non-linearities and thus new couplings which are uncontrolled i.e. there is no reason why these should now be irrelevant. The second obvious result is that equation (2.52) is no longer valid as a leading order result; the series for this now diverges. Later, Yakhot & Orszag along with other authors attempted to get past the problems of extra non-linearities by introducing a *distant interaction approximation* in which the high and low wavenumber modes are assumed to interact in a non-local fashion and thus the higher order non-linearities, which represent local interactions, are neglected.

- Secondly *any* renormalized viscosity should not depend upon our particular choice of the forcing exponents. Even in an LES the effective viscosity should not behave as if the forcing *in all cases* is given by equation (2.42) with the exponent $y = 3$. In fact the scale-free inertial range with the energy spectrum given by the Kolmogorov $k^{-5/3}$ form is independent of forcing and viscous effects. Yakhot and Orszag get around this by introducing their *'correspondence principle'* which states that the FNS forcing terms and statistics are *statistically equivalent* to the boundary and initial conditions of a freely cascading turbulent flow with the forcing chosen correctly.

With these problems aside, we write equations (2.51) and (2.52) as

$$\nu^*(k) \simeq \left(\frac{D^0}{20}\right)^{1/3} k^{-4/3} \,, \tag{2.53}$$

and

$$E(k) \simeq \left(\frac{20 S_3^2}{(2\pi)^6} \left(D^0\right)^2\right)^{1/3} k^{-5/3} \,. \tag{2.54}$$

Yakhot and Orszag then relate the parameter D^0 to the energy dissipation rate ε by equating equations (2.53) and (2.54) with the inertial range forms $E(k) \simeq \alpha \varepsilon^{2/3} k^{-5/3}$ and $\nu(k) = \beta \varepsilon^{1/3} k^{-4/3}$ to get

$$D^0 = \frac{(2\pi)^3}{2 S_3} \left(\frac{\alpha}{1.186}\right)^{3/2} \varepsilon \,, \tag{2.55}$$

and

$$D^0 = 20 \beta^3 \varepsilon \,. \tag{2.56}$$

Using an analytical single-time Markovianized model based on Kraichan's DIA second order closure (renormalized perturbation theory), Yakhot and Orszag use the relation (see McComb [5] p.257)

$$\beta = 0.19 \alpha^2 \,, \tag{2.57}$$

to obtain the prediction $\alpha = 1.617$, which along with the other parameters they estimate from their theory, is in very good agreement with experiments. In the end, these are the very results that Yakhot & Orszag use to justify their approximations; the logic being that good results imply that there must be something correct about the theory.

2.2.4 Comments

Many authors [47, 48, 49, 50, 51] that used the ϵ-expansion variant of RG, and who followed afterwards, were mainly embroiled with trying to understand and apply the work of Yakhot & Orszag, as they were the first who had tried to apply the work of FNS to actual turbulence. Application of their theory was extended into turbulence modeling such as Reynolds stress, $K - \varepsilon$ and LES models, as well as to the study of related systems. A few authors occupied

themselves with the harder and higher priority task of trying to clean the theory and sort out the major problems therein. Amongst the latter are the works of Smith and Reynolds [52] and Smith and Woodruff [53], as well as the recent work of Sukoriansky *et. al.* [54] who emphasise the need to deal with higher order cross-term non-linearities. Of particular note is the work of Lam [55] and Tomassini [56] who showed the variation of the predictions for the turbulent parameters when the statistics of the forcing are changed. The use of results from another theory, especially from the class of RPT closures with unknown convergence properties, is another problematic factor. Maybe the good results that Yakhot & Orszag obtain are just symptomatic of these theories. Kraichnan [57] picked up on this problem, and showed that many of the results of Yakhot & Orszag can be obtained from a simple perturbation based treatment using the same assumptions and not with any RG techniques at all. Eyink [42] emphasised this latter point and argued, much like in this thesis, that the work of Yakhot and Orszag was not RG at all. Eyink also claims that the higher order non-linearities are marginal and not irrelevant, even in the case of the distant interaction approximation. McComb comments on the unphysical nature of the choice of $\epsilon = 4$ and the correspondence principle, as from equation (2.42) it introduces a energy input rate $\propto k^{-1}$ which diverges at the origin.

Indeed one would be safe in saying that the work of Yakhot & Orszag has attracted as much controversy as interest. The daring approximations that Yakhot & Orszag introduced, go completely against the grain of the approximation controlling that RG is particularly successful in. All of the above mentioned work and its extensions by other authors has to ultimately come to terms with the fact that all their approximations are only valid in the limit $k \to 0$, and hence so are any results thereof.

2.3 Recursive RG

What we are loosely labeling as ‘recursive RG’ is a set of theories in which finite shells of degrees of freedom are removed. The recursion relations are in the form of iteration maps taking the renormalized parameters from one coarse-grained picture to the next. This approach was first introduced by Rose (1977) [58] around the same time as FNS. However, Rose’s method was not applied to the NSE. Rose was studying passive scalar transport in an artificial

'toy problem' of a passive scalar contaminant advected by a random and frozen (in time) velocity field. This will be discussed in more detail in Chapter 6. The crucial difference between this method and the ones mentioned above is that there is no approximation which requires a $k \to 0$ limit. In fact the authors of these methods start off in the far UV part of the energy spectrum and by removing finite shells using RG, stop at the fixed point which corresponds to the inertial range Kolmogorov scaling. The method also has no ϵ-expansion, as there is no restrictions on what form or statistics the forcing, which sustains the flow, takes. Thus although an expansion is employed, it is not around a Gaussian zero order field. The expansion of the velocity field is obtained in terms of an iterative use of the NSE itself.

2.3.1 McComb *et. al.* (1982)

The 'iterative averaging' work of McComb *et. al.* [59, 60] was the earliest attempt to apply the Rose type RG to fully developed turbulence. It was later reformulated in a form in which the averaging procedure was given a detailed study [61, 62], and later in [63, 64]. In particular, emphasis was placed on the filtered ensemble average used in the FNS study. The filtered ensemble average is needed for averaging over small scales whilst the large scales are held fixed. McComb argued that although this is possible for microscopic physics, the corresponding principle for classical dynamical systems is impossible in principle. This is because if you fix large scales to a preferred realisation then the deterministic nature of the NSE implies that you will also fix the small scales and thus any form of averaging will have no effect since you only have one member of the ensemble. To overcome this problem McComb *et. al.* introduced a *weak conditional average* which allows the construction of an ensemble. The evaluation of an average over this conditioned ensemble is then done by use of an asymptotic freedom (in wavenumber) approximation. Much of this thesis will be concerned with this method and discussing in detail the approximations therein. Without too much repetition however, we will summarise here some of the key steps and results.

We start with equation (2.4) and apply the conditional average (CA) denoted by $\langle\ \rangle_c$, and which is an average over a sub-ensemble of the turbulent velocity

field realisations that all have low-k velocity mode realisations given by

$$u_k^- + \phi_k^- \ , \tag{2.58}$$

where ϕ_k^- is a random function which is introduced to generate the ensemble, and can be made as small as we want. The CA has the property

$$\begin{aligned} \left\langle u_k^- \right\rangle_c &= u_k^- + \left\langle \phi_k^- \right\rangle_c \\ &= u_k^- \ , \end{aligned} \tag{2.59}$$

where the result in the second line follows from the requirement that

$$\left\langle \phi_k^- \right\rangle_c = 0 \ . \tag{2.60}$$

With this CA we have the following

$$\begin{aligned} \left\langle u_k^- u_j^- \right\rangle_c &= u_k^- u_j^- + 2u_k^- \left\langle \phi_j^- \right\rangle_c + \left\langle \phi_k^- \phi_j^- \right\rangle_c \\ &\simeq u_k^- u_j^- \ , \end{aligned} \tag{2.61}$$

as an example; where in the second line we have used (2.60) and $\left\langle \phi_k^- \phi_j^- \right\rangle_c \simeq 0$. We also see that

$$\begin{aligned} \left\langle u_k^- u_j^+ \right\rangle_c &= u_k^- \left\langle u_j^+ \right\rangle_c + 2 \left\langle u_j^+ \phi_j^- \right\rangle_c + \left\langle \phi_k^- \phi_j^- \right\rangle_c \\ &\simeq u_k^- \left\langle u_j^+ \right\rangle_c \ , \end{aligned} \tag{2.62}$$

where the second line follows from the small nature of ϕ. We are now left with trying to evaluate $\left\langle u_j^+ \right\rangle_c$. McComb *et. al.* tackle this by introducing the *asymptotic freedom approximation*. This states that if we evaluate the CA $\left\langle u_j^+ \right\rangle_c$ at the UV cut-off k_0, far from the partitioning wavenumber k_c we can expect that the CA subensemble begins to resemble the full unrestricted ensemble average, provided that the shell being averaged over is large enough. This is so because the effect of the factor ϕ increases, due to the chaotic nature of the non-linearity, as we move through k-space away from k_c towards k_0, thus decoupling the modes at these wavenumbers and generating an ensemble similar to the full ensemble. The asymptotic freedom approximation can then be said to approximate the CA such that it can be evaluated at the UV cut-off k_0 as a full ensemble average. This will be explained in much more clarifying

detail in the next chapter where the concept of a CA is given a functional based treatement. This being so, we may now make the approximation

$$\left\langle u_j^+ \right\rangle_c \simeq \lim_{j \to k_0} \left\langle u_j^+ \right\rangle \ , \tag{2.63}$$

which for this case will result in zero. Taking a CA on equation (2.4), then gives the result

$$\left[\frac{\partial}{\partial t} + \nu^0 k^2 \right] u_k^- = \lambda_0 M_k^- \left[u_j^- u_{k-j}^- + \left\langle u_j^+ u_{k-j}^+ \right\rangle_c \right] \ . \tag{2.64}$$

The CA of the Reynolds like term is then handled by constructing a dynamical equation for $u_j^+ u_{k-j}^+$ from the NSE, apply the CA on it and evaluate using the approximation of asymptotic freedom. Then substitute the result into equation (2.64), which results in a RG recursion relation

$$\nu_1(k) = \nu_0 + \delta\nu_0(k) \ , \tag{2.65}$$

with an increment to the viscosity given by

$$\delta\nu_0(k) = \frac{1}{k^2} \int d^3 j \frac{L(\mathbf{k},\mathbf{j}) \left[Q(|\mathbf{k}-\mathbf{j}|)|_{|\mathbf{k}-\mathbf{j}|=k_0} + (|\mathbf{k}-\mathbf{j}| - k_0) \left. \frac{\partial Q(|\mathbf{k}-\mathbf{j}|)}{\partial |\mathbf{k}-\mathbf{j}|} \right|_{|\mathbf{k}-\mathbf{j}|=k_0} \right]}{\nu_0 j^2 + \nu_0 \left| \mathbf{k}-\mathbf{j} \right|^2} \ , \tag{2.66}$$

where Q is the spectral density function and

$$L(\mathbf{k},\mathbf{j}) = \frac{\left[\mu \left(k^2 + j^2 \right) - kj \left(1 + 2\mu^2 \right) \right] \left(\mu^2 - 1 \right) kj}{k^2 + j^2 - 2kj\mu} \ , \tag{2.67}$$

where μ is the cosine of the angle between the wavevectors $\mathbf{k}$ and $\mathbf{j}$. When equations (2.65) and (2.66) are written in dimensionless form and rescaling is taken into account, iteration of the equations gives a scale invariant fixed point renormalized viscosity. Using the definition of the dissipation rate given by equation (1.37) McComb *et. al.* then go on to obtain a value for the Kolmogorov constant which they calculate to be $\alpha = 1.62$, in excellent agreement with experimental results.

The work of McComb *et. al.* has the advantage that it is considerably cleaner than most of the other RG approaches, especially those of the Yakhot & Orszag type, and thus any approximations are quite clearly presented. It has also been applied to spectral LES with results being as good as any others in the field. Problems associated with this scheme will be discussed in Chapter 5.

2.3.2 Zhou, Vahala and Hossain (1988)

The last of the RG schemes that we will discuss in this chapter is that by Zhou *et. al*. (ZVH) [65]. Although ZVH have applied their theory to both forced and freely decaying turbulence, we will only be discussing their results for free decay. Their forced NSE theory uses similar assumptions to Yakhot & Orszag but without any ϵ-expansion. They use a filtered ensemble average in a similar way to FNS and the earlier works of McComb [60]. They start with applying the filtered average on equation (2.4) to get

$$\left[\frac{\partial}{\partial t}+\nu^0 k^2\right] u_k^- = \lambda_0 M_k^- \left[u_j^- u_{k-j}^- + 2u_j^- \left\langle u_{k-j}^+ \right\rangle_{high} + \left\langle u_j^+ u_{k-j}^+ \right\rangle_{high}\right] , \quad (2.68)$$

The cross term is then handled by substituting the formal solution of equation (2.5), and the Reynolds term is substituted for in a similar way. This results in an eddy damping term η, and also the introduction of a cubic non-linearity term so that equation (2.68) becomes

$$\begin{aligned}\left[\frac{\partial}{\partial t}+\nu^0 k^2\right] u_k^- + \eta_k^0 u_k^- &= \lambda_0 M_k^- u_j^- u_{k-j}^- \\ &\quad +\lambda_0^2 2 M_k^- M_j^+ G_j^0 u_p^- u_{j-p}^- u_{k-j}^- + \mathcal{O}\left(\lambda_0^3\right) . \quad (2.69)\end{aligned}$$

This is the equation which is then used in the formal RG calculation; ZVH arguing that a cubic non-linearity needs to be retained in a similar way to the retention of next-to-neighbour interactions in the RG application to the Ising model. This results in an RG recursion relation given by equation (2.65) but with the n^{th} increment to the viscosity given by

$$\delta\nu_n(k) = 2\sum_{i=0}^{n}\int d^3 j \frac{L(\mathbf{k},\mathbf{j})Q(|\mathbf{k}-\mathbf{j}|)}{\nu_i(j) j^2 k^2} . \quad (2.70)$$

Iterating to a fixed point gives the ZVH fixed point renormalized scale-invariant viscosity. This leaves the reduced dynamical equation looking like

$$\left[\frac{\partial}{\partial t}+\nu_k^D k^2\right] u_k = M_k u_j u_{k-j} + 2M_k M_j \sum_i^n G_j^i u_p u_{j-p} u_{k-j} , \quad (2.71)$$

where ν_k^D is the renormalized viscosity, and all superscript $-$'s have been dropped. ZVH then handle the extra terms from the cubic non-linearity by bolt-

ing it onto the renormalized viscosity. This is done by constructing the energy balance from equation (2.71) in a similar way to how equation (1.30) was derived. This results in the cubic term turning into a quartic term which ZVH then factorise into products of second order moments using the quasi-normal approximation of Proudman and Reid[15] (see Chapter 7), resulting in an energy balance equation[4] looking like

$$\frac{\partial Q_k}{\partial t} = -2\nu_k^D k^2 Q_k + T_k + \sum QQ \ . \tag{2.72}$$

The second order product terms $\sum QQ$ are then written as

$$\sum QQ = -2\nu_k^T k^2 Q_k \ , \tag{2.73}$$

thus resulting in an enhancement to the already renormalized viscosity to produce an effective viscosity

$$\nu_k^{net} = \nu_k^D + \nu_k^T \ . \tag{2.74}$$

ZVH argue that the introduction of a cubic non-linearity is necessary in producing some of the essential features of a correct effective viscosity. The form of their effective viscosity agrees qualitatively well with the form produced from numerical simulations, especially with the production of a strong cusp near $k = k_c$. However, the effective viscosity of ZVH introduces a problem. This is that although ZVH manage to produce a cusp, they had to resort to using a production spectra (see Leslie & Quarini [66]) to handle a divergence that appears as $k \to k_c$; the cusp is a result of taming this divergence by using a realistic energy spectrum which goes to zero as $k \to 0$, instead of using the Kolmogorov energy spectrum which diverges in this limit. The intensity of the cusp is strongly dependent upon the choice of model parameters used to construct this production spectrum and hence does not exhibit universal and scale-free behaviour, although part of it does. The phenomenon of a cusp being a result of a divergence and then being controlled by some method which relies on arbitrary parameters was also seen in Kraichnan's Test Field Model [67], and is equivalent to introducing a cut-off in the limit of the divergent integrals. This topic of cusps and divergences will be discussed later in Chapter 5.

[4] We have used the spectral energy density here, instead of the equivalent $E(k)$ form, as this is how ZVH originally present their work.

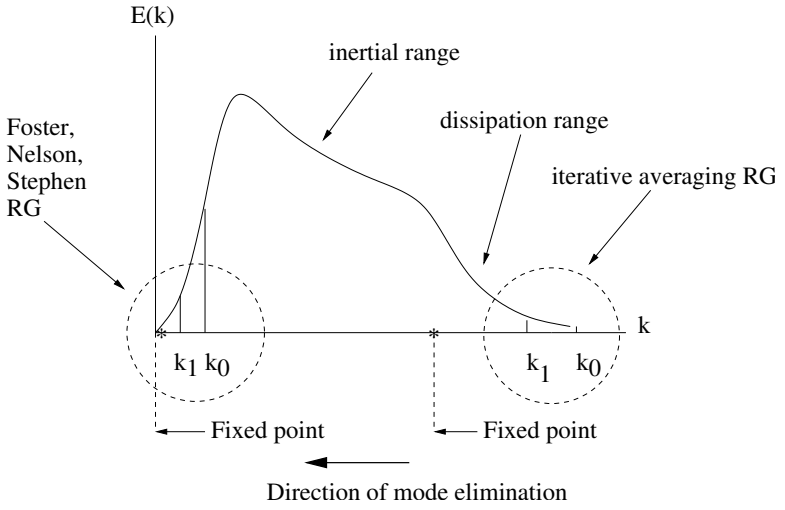

Figure 2.2: Schematic illustrating the smallness of the local Reynolds number in the IR and UV asymptotic regions and the fixed points arising from the two characteristic RG schemes that work in these different regions.

2.4 Discussion

Throughout this review we have seen that all the RG schemes end up using a variant of perturbation theory with the expansion parameter being the reduced coupling represented by the local Reynolds number. The local Reynolds number can be written in a general and representative way by (see Batchelor [68])

$$R(k_c) \simeq \frac{1}{\nu}\left(\frac{E(k_c)}{2\pi k_c}\right) \ . \tag{2.75}$$

For a perturbation series to be useful we need this parameter to be kept small so that some sort of lower order truncation is possible. From equation (2.75) we can see that $R(k_c)$ is only small in the regions where $E(k) \to 0$, which only happens in the two asymptotic regions of $\lim k \to 0$ and $\lim k \to \infty$, i.e. the IR and UV asymptotic regions. The FNS ϵ-expansion type schemes all seem to work in the IR region, and the McComb iterative averaging like schemes work in the UV region. The associate fixed points for these two regions are shown in figure 2.2, which also serves as a good summary illustrating the regions of validity of the RG schemes discussed in this chapter.

Chapter 3

A functional representation of the conditional average

Crucial to the statistical study of turbulence is the notion of averaging over an ensemble of velocity fluctuation realisations. The generation of this ensemble was discussed earlier in the introduction. Here, we will be discussing a particular form of ensemble averaging that is needed in order to do the RG procedure - conditional ensemble averaging. We will begin by reviewing the construction of the ensemble average in a functional representation by introducing the concept of a state space where each basis is formed from the different realisations comprising the ensemble. In the previous chapter the dynamical RG method was introduced and some applications of it to the study of isotropic turbulence were mentioned. The most important non-trivial step outlined in dynamical RG was the process of coarse-graining or the averaging out of degrees of freedom; this being the step distinguishing one RG method from another. We will discuss the need for a conditional average in the coarse-graining of such systems. From this we will proceed to extending the functional formulation of the ensemble average to the case of the conditional average, and illustrating its use by applying it to some simple cases. We will finish by providing a derivation of this functional conditional average via the introduction of the *conditional projector* and *asymptotic freedom* operations, and we will show that the conditional average used earlier was just one of many possible representations.

3.1 Ensemble averaging

The expectation value of a function of a random variable y with a probability distribution $P(y)$, is given by

$$\langle f\,(y)\rangle = \int_R dy P(y) f\,(y)\,, \tag{3.1}$$

where R is the region of y over which the expectation value is taken, and the probability distribution is assumed properly normalised[1]

$$\int_R dy P(y) = 1. \tag{3.2}$$

Equation (3.1) returns the average value of the function $f(y)$ over R.

Before we move onto taking more complicated averages involving ensembles of realisations of random functions, we must take a precaution on notation.

Firstly, we will be aiming to average a random function $u(x)$, say, inside a partial differential equation which determines its behaviour e.g.

$$\frac{\partial u(x)}{\partial x} + \frac{\partial u^2(x)}{\partial x} = \frac{\partial^2 u(x)}{\partial x^2}. \tag{3.3}$$

In (3.3), $u(x)$ represents a single realisation from an ensemble. To avoid any confusion in the notation we will represent the ensemble of realisations by the set $\lim_{n\to\infty}\{w_1(x), w_2(x)\ldots w_n(x)\}$, which we shall denote $\{w(x)\}$ to save space, such that $u(x) \in \{w(x)\}$. One should note that the set here is assumed to be a continuous set, such that there exists an infinite number of realisations[2]. The expectation value, (3.1), differs from this because it involves an ensemble of random *variables*, $\{y\}$, where in the present case we are now considering an ensemble of random *functions*. Apart from this additional complication they are essentially similar.

When we talk of ensemble averaging equation (3.3), we can look at this as applying an operation on it such that it replaces any function (or functional; see

[1] If it were not normalised then we would simply incorporate the division of (3.1) by (3.2) into the definition of the ensemble average.

[2] Readers interested in the aspects pertaining to the details of continuity in the realisations and other functions and functionals in this chapter, please refer to Beran [69] and Beerends [70].

later) of $u(x)$ by its average value. As in (3.1) we will denote the operation of this type of averaging by angle brackets $\langle\ \rangle$.

Secondly, although mentioned in more detail in Appendix A, a reminder should be made on what a functional is:

> A functional $F[u(x)]$ is a quantity which depends continuously upon all the values a function $u(x)$ takes in some interval $a \leq x \leq b$.

Thus, $F[u(x)] = \int_a^b u^2(x)dx$ would be considered a functional; whereas $f(u(x)) = u^2(x)$ would not be. In the former, x is not really important because it is a dummy variable whilst in the latter this is not the case. Thus, for the former it is $u(x)$ which is of primary importance as an argument, whilst in the latter it is x. Another example of a functional is the expectation value average given by (3.1).

We can now generalise the result in (3.1) to taking the expectation value of a function f which depends on the single value a random function $u(x)$ takes at some point x_i

$$\langle f\left(u(x_i)\right)\rangle = \int dw(x_i) P_1\left(w(x_i)\right) f\left(w(x_i)\right), \tag{3.4}$$

again with the normalisation condition

$$\int dw(x_i) P_1\left(w(x_i)\right) = 1. \tag{3.5}$$

This gives the expectation value of the function $f\left(u(x_i)\right)$ over all the realisations or *paths* $\{w(x)\}$ at a single point x_i, where $P_1\left(w(x_i)\right)$ is the single point probability distribution of the path $w(x)$ at x_i. The generation of this (and later) ensembles is shown in Figure 3.1. Heuristically, (3.4) is really just the same as (3.1) if we rename $y = u(x_i)$ and $y = w(x_i)$.

If we were now to take the expectation value of the function f such that it depends upon the two values $u(x)$ takes at points x_1 and x_2, we would then have

$$\langle f\left(u(x_1), u(x_2)\right)\rangle = \int dw(x_1)dw(x_2) P_2\left(w(x_1), w(x_2)\right) f\left(w(x_1), w(x_2)\right), \tag{3.6}$$

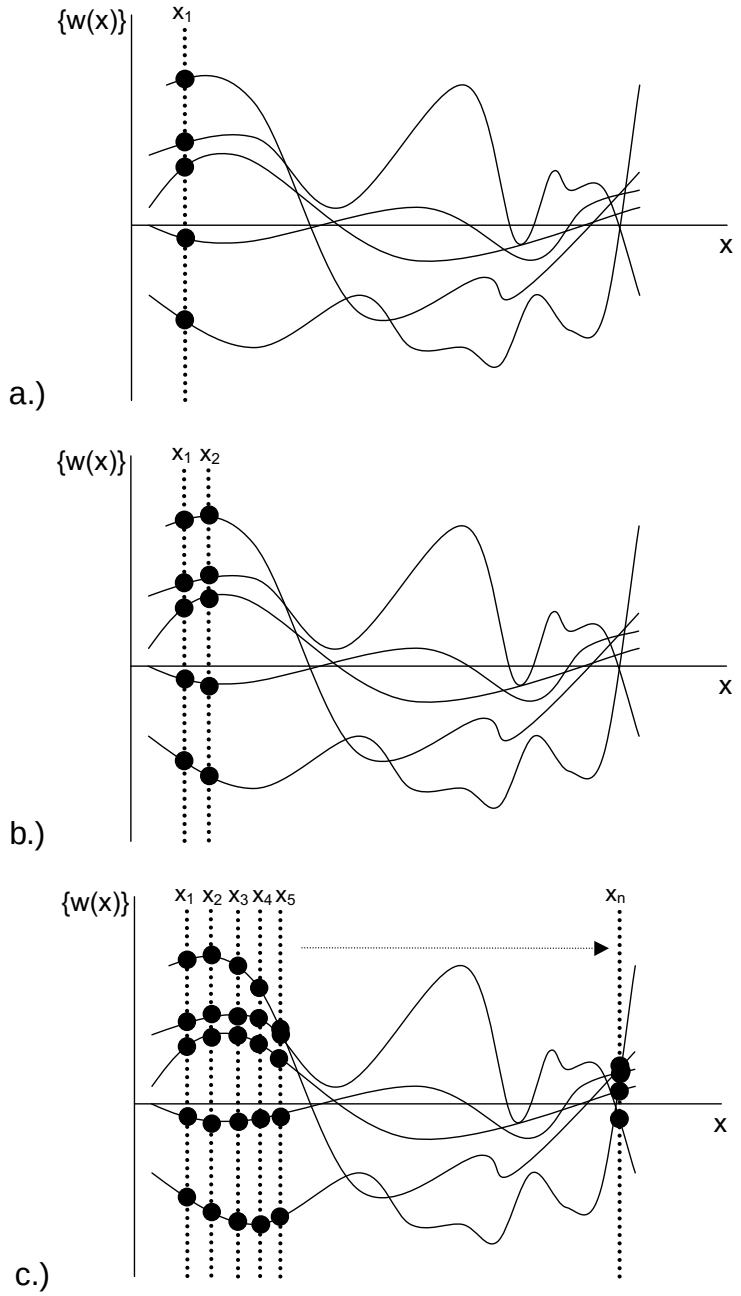

Figure 3.1: Schematic showing the selection of points of a random variable $u(x)$ for use in ensemble averaging. We use a set of five realisations, labeled as $\{w(x)\}$, from the ensemble as an example. Illustrations are for the case of a.) selection at, x_1, generating the ensemble for $u(x_1)$ (by picking the value of $u(x_1)$ for each realisation), b.) for x_1 and x_2 generating the ensemble for $u(x_1)$ and $u(x_2)$ and c.) for $x_1 \dots x_n$ generating the ensemble for n random variables $u(x_1), \dots, u(x_n)$. It is important to remember that in our functional formalism each of the points making the ensemble has a probability weight associated with it.

which returns the average value of the function $f\left(u(x_1), u(x_2)\right)$ over all the realisations/paths $\{w(x)\}$ at the two points x_1 and x_2, where the two-point probability distribution involved is assumed properly normalised. Again, this can be seen, heuristically, as an expectation value of the two-dimensional function $f(x, y)$ if we rename $x = u(x_1)$ and $y = u(x_2)$.

We can extend this result such that an expectation value of the function f depending upon the n values $u(x)$ takes at points $x_1, x_2, \ldots x_n$, would be

$$\begin{aligned}\left\langle f\left(u(x_1), u(x_2) \ldots u(x_n)\right)\right\rangle &= \int dw(x_1)dw(x_2)\ldots dw(x_n) \times \\ &\times P_n\left(w(x_1), w(x_2)\ldots w(x_n)\right) \times \\ &\times f\left(w(x_1), w(x_2)\ldots w(x_n)\right). \end{aligned} \tag{3.7}$$

More generally, for $n \leq m$ such that $\{u_1 \ldots u_n\} \subset \{u_1 \ldots u_m\}$, we could instead write

$$\begin{aligned}\left\langle f\left(u(x_1), u(x_2) \ldots u(x_n)\right)\right\rangle &= \int dw(x_1)dw(x_2)\ldots dw(x_n), dw(x_{n+1})\ldots dw(x_m) \\ &\quad \times P_m\left(w(x_1), w(x_2)\ldots w(x_n), w(x_{n+1})\ldots w(x_m)\right) \\ &\quad \times f\left(w(x_1), w(x_2)\ldots w(x_n)\right) \\ &= \left\langle f\left(u(x_1), u(x_2) \ldots u(x_n)\right)\right\rangle \\ &\times \int dw(x_{n+1})\ldots dw(x_m) P_{m-n}\left(w(x_{n+1})\ldots w(x_m)\right) \\ &= \left\langle f\left(u(x_1), u(x_2) \ldots u(x_n)\right)\right\rangle, \end{aligned} \tag{3.8}$$

where one should note that in the last step the probability distribution has been changed from P_m to P_{m-n}. It is important to distinguish between these two probability distributions as otherwise we would be assuming (implicitly) that the distributions can be factorised and retain form invariance. We have no information about the distribution so we cannot assume this. In fact this is only generally true for Gaussian distributions, and we know that for problems with coupling such as turbulence, P_m is non-Gaussian. Finally, in the last step one has again used the property of the probability distributions being assumed properly normalised

$$\int dw(x_{n+1})\ldots dw(x_m) P_{m-n}\left(w(x_{n+1})\ldots w(x_m)\right) = 1. \tag{3.9}$$

We are now ready to take the continuum limit such that

$$\lim_{n\to\infty} |u(x_{n+1}) - u(x_n)| = 0, \tag{3.10}$$

and the functions in (3.7) become functionals and the integral becomes a functional integral

$$\langle f\left[u\left(x\right)\right]\rangle = \int_R \mathcal{D}w\left(x\right) P_R\left[w\left(x\right)\right] f\left[w\left(x\right)\right], \tag{3.11}$$

along with the normalisation condition

$$\int_R \mathcal{D}w\left(x\right) P_R\left[w\left(x\right)\right] = 1, \tag{3.12}$$

where R is the region over which the functional integration is defined and P_R is the probability distribution functional which depends continually on all values that the function as its argument ($w(x)$) takes on R. The equivalent of (3.8) would be

$$\langle f\left[u\left(x\right)\right]\rangle = \int_{R'} \mathcal{D}w\left(z\right) P_{R'}\left[w\left(z\right)\right] f\left[w\left(x\right)\right], \tag{3.13}$$

where R' is a greater region than R such that $R \subset R'$, and z takes all values in R' whilst x takes all values in R.

We can generalise this result to vector fields and functions of more than one variable by first considering the space of the variables taking only two points e.g. the variables and indices in $u_\alpha(x,t)$ take the values $\alpha = \{1,2\}$, $x = \{x_1, x_2\}$ and $t = \{t_1, t_2\}$, such that the random variables considered are

$$\begin{array}{ll} u_1(x_1,t_1), & u_2(x_1,t_1), \\ u_1(x_1,t_2), & u_2(x_1,t_2), \\ u_1(x_2,t_1), & u_2(x_2,t_1), \\ u_1(x_2,t_2), & u_2(x_2,t_2). \end{array}$$

In this discrete form, the average of a function of all these variables would be written as

$$\begin{aligned} \langle f\left(u_1(x_1,t_1),\cdots,u_2(x_2,t_2)\right)\rangle \;=\; & \int dw_1(x_1,t_1),\cdots,\int dw_2(x_2,t_2) \\ & \times \widetilde{P}\left(w_1(x_1,t_1),\cdots,w_2(x_2,t_2)\right) \times \\ & \times f\left(w_1(x_1t_1),\cdots,w_2(x_2,t_2)\right), \end{aligned} \tag{3.14}$$

where $\widetilde{P}$ is the probability distribution for this specific case. If one now increases the number of points and takes the continuum limit in the variables x and t in a similar way to (3.10), we get to the result[3]

$$\langle f\left[u_\alpha\left(x,t\right)\right]\rangle = \int_R \mathcal{D}w_\alpha\left(x,t\right) P_R\left[w_\alpha\left(x,t\right)\right] f\left[w_\alpha\left(x,t\right)\right], \tag{3.15}$$

where α, for future convenience, takes the Cartesian values 1, 2 and 3.

Similar arguments can extend this further to functionals of many functions of scalar or vector arguments e.g. spanning domains S and R with a probability distribution functional P_{SR}, say

$$\begin{aligned}\langle f\left[u_\alpha\left(\mathbf{x},t\right),u_\beta\left(\mathbf{y},t\right)\right]\rangle &= \int_S\int_R \mathcal{D}w_\alpha\left(\mathbf{x},t\right)\mathcal{D}g_\beta\left(\mathbf{y},t\right) P_{SR}\left[w_\alpha\left(\mathbf{x},t\right),g_\beta\left(\mathbf{y},t\right)\right]\times \\ &\quad \times f\left[w_\alpha\left(\mathbf{x},t\right),g_\beta\left(\mathbf{y},t\right)\right],\end{aligned} \tag{3.16}$$

along with the normalisation condition

$$\int_S\int_R \mathcal{D}w_\alpha\left(\mathbf{x},t\right)\mathcal{D}g_\beta\left(\mathbf{y},t\right) P_{SR}\left[w_\alpha\left(\mathbf{x},t\right),g_\beta\left(\mathbf{y},t\right)\right] = 1. \tag{3.17}$$

(3.16) gives the mean or expected functional of $f\left[u_\alpha\left(\mathbf{x},t\right),u_\beta\left(\mathbf{y},t\right)\right]$; the mean of the functional depending upon the entire domains S and R spanned by the functions $u_\alpha\left(\mathbf{x},t\right)$ and $u_\beta\left(\mathbf{y},t\right)$ respectively.

The above formulation of the ensemble average has resulted in the average of a functional, i.e. a function that takes as its argument a function over a certain domain. In most cases we will be averaging functions and not functionals. Because of the generality of the above formulation, this is easily accommodated as a particular case. Thus, the average of a function $f\left(u_\alpha\left(\mathbf{x},t\right)\right)$is given by

$$\langle f\left(u_\alpha\left(\mathbf{x},t\right)\right)\rangle = \int_R \mathcal{D}w_\lambda\left(\mathbf{z},\tau\right) P\left[w_\lambda\left(\mathbf{z},\tau\right)\right] f\left(w_\alpha\left(\mathbf{x},t\right)\right), \tag{3.18}$$

where this will result in the particular values of α, $\mathbf{x}$ and t being picked out from the combinations taken by λ, $\mathbf{z}$ and τ in the functional integration. Also note that we have dropped the R subscript on the probability distribution functional P. This form is analogous to (3.8) and (3.13) except that the $\mathbf{x}$, t and α in this case are at a particular value rather than over a domain.

[3]Please note that the P_R probability distribution here is not the same as the one in (3.11).

3.2 The need for a conditional average

Let us suppose that we want to solve for n components of a complex coupled m component system, where $n < m$, without solving the full system. In general the n components we want to solve for will be dependent, through the strength of some coupling parameters, on all m components. The simplest approximation that one can make is a mean-field one in which a conditional expectation is evaluated which holds the n components constant whilst averaging over the remaining $m - n$ components [71]. This is exactly the situation facing anyone trying to coarse-grain a system, as is the case in a RG calculation.

Translating this into our system for the NSE, we seek to coarse-grain our system in terms of the low-wavenumber modes u^- only. Most authors in the field, notably FNS, Yakhot & Orszag, ZVH and their variants, use a filtered ensemble average which holds the u^- constant whilst averaging, using a full ensemble of turbulent velocities, over the u^+ modes. The work of McComb *et. al.* strongly disagrees with this approach, and argues that, in principle, this is impossible to do as the deterministic nature of the NSE will imply that if you hold the u^- modes constant then you will also be holding the u^+ modes constant too. Instead what is needed is a conditional average (CA) much like a filtered ensemble average but with the constraint on the CA of holding the u^- modes constant, being weakened. The introduction of this *weak conditional average* was introduced by McComb, Watt and Roberts [61] and later developed by McComb and Johnston [63], as mentioned earlier in the previous chapter. The CA in the latter works was applied in a discrete ensemble sum averaging formulation. In the present work we aim to recast this in a functional representation, and in particular introduce the dealing with conditional probability distributions.

It is also worth noting that Eyink [42] also made a similar criticism as McComb, of the FNS constraints on the averaging procedure but with more of an emphasis on the forcing statistics. This criticism was later tackled by Hunter [72].

3.3 Conditional Averaging

We now want to represent a functional form for the conditional average i.e an expectation value conditioned/biased on a sub-ensemble (see [61, 63, 64]). In

particular we require properties such as:-

$$\widehat{\mathcal{P}}_c^- \left\{ u_\alpha^- (\mathbf{k},t)\, u_\beta^- (\mathbf{j},t) \ldots u_\gamma^- (\mathbf{l},t) \right\} = u_\alpha^- (\mathbf{k},t)\, u_\beta^- (\mathbf{j},t) \ldots u_\gamma^- (\mathbf{l},t) \quad (3.19)$$

$$\widehat{\mathcal{P}}_c^- \left\{ u_\alpha^- (\mathbf{k},t)^n \right\} = u_\alpha^- (\mathbf{k},t)^n \quad (3.20)$$

$$\widehat{\mathcal{P}}_c^- \left\{ u_\alpha^- (\mathbf{k},t) \ldots u_\gamma^- (\mathbf{l},t)\, u_\lambda^+ (\mathbf{m},t) \ldots u_\delta^+ (\mathbf{p},t) \right\} = u_\alpha^- (\mathbf{k},t) \ldots u_\gamma^- (\mathbf{l},t)\, \widehat{\mathcal{P}}_c^- \left\{ u_\lambda^+ (\mathbf{m},t) \ldots u_\delta^+ (\mathbf{p},t) \right\} \quad (3.21)$$

where $\widehat{\mathcal{P}}_c^-$ denotes the conditional projection operator biased on u^- modes. We suggest a form for this projector by construction, and its application is given by

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Definition

$$\widehat{\mathcal{P}}_c^- \left\{ f\left[u_\alpha (\mathbf{k},t) \right] \right\} = \frac{1}{N!} \prod_{i=1}^{N} \int d^3 p_i \int ds_i u_{\sigma_i}^- (\mathbf{p}_i, s_i) \int \mathcal{D} w_\lambda (\mathbf{z},\tau) \times \times P\left[w_\lambda (\mathbf{z},\tau) \right] \frac{\delta^N f\left[w_\alpha (\mathbf{k},t) \right]}{\delta w_{\sigma_1}^- (\mathbf{p}_1, s_1) \ldots \delta w_{\sigma_N}^- (\mathbf{p}_N, s_N)}, \quad (3.22)$$

---

where the condition, in this case, is over a sub-ensemble biased on resolved or low-wavenumber turbulent velocity amplitudes/modes. The derivation of this form of the functional conditional average is provided in the penultimate section. This operation has the desired effect of leaving the low k-filtered members of the biased sub-ensemble unaveraged whilst averaging all other members of the ensemble in the normal way (see section 3.1). N indicates the number of low-wavenumber modes that $f\left[u_\alpha (\mathbf{k},t) \right]$ contains. It will be shown below that even if the function does not contain any apparent low-wavenumber modes it can be expanded into a power series of such modes using the NSE.

We will now prove that (3.22) does indeed give equalities (3.19)-(3.21). Beginning with (3.19) and using n instead of the N in (3.22) we obtain

$$\widehat{\mathcal{P}}_c^- \left\{ u_{\alpha_1}^- (\mathbf{k}_1,t) \ldots u_{\alpha_n}^- (\mathbf{k}_n,t) \right\} = \frac{1}{n!} \prod_{i=1}^{n} \int d^3 p_i \int ds_i u_{\sigma_i}^- (\mathbf{p}_i, s_i) \times \times \int \mathcal{D} w_\lambda (\mathbf{z},\tau)\, P\left[w_\lambda (\mathbf{z},\tau) \right] \times \times \frac{\delta^n w_{\alpha_1}^- (\mathbf{k}_1,t) \ldots w_{\alpha_n}^- (\mathbf{k}_n,t)}{\delta w_{\sigma_1}^- (\mathbf{p}_1, s_1) \ldots \delta w_{\sigma_n}^- (\mathbf{p}_n, s_n)}; \quad (3.23)$$

now focusing on the functional derivatives inside the braces on the RHS we have

$$\begin{aligned}\frac{\delta^n w^-_{\alpha_1}(\mathbf{k}_1,t)\dots w^-_{\alpha_n}(\mathbf{k}_n,t)}{\delta w^-_{\sigma_1}(\mathbf{p}_1,s_1)\dots\delta w^-_{\sigma_n}(\mathbf{p}_n,s_n)} &= \frac{\delta^{n-1}}{\delta w^-_{\sigma_2}(\mathbf{p}_2,s_2)\dots\delta w^-_{\sigma_n}(\mathbf{p}_n,s_n)}\times\\ &\times\Big[\,\delta_{\alpha_1\sigma_1}\delta(\mathbf{k_1}-\mathbf{p_1})\delta(t-s_1)w^-_{\alpha_2}(\mathbf{k}_2,t)\dots w^-_{\alpha_n}(\mathbf{k}_n,t)+\\ &\quad+w^-_{\alpha_1}(\mathbf{k}_1,t)\,\frac{\delta w^-_{\alpha_2}(\mathbf{k}_2,t)\dots w^-_{\alpha_n}(\mathbf{k}_n,t)}{\delta w^-_{\sigma_1}(\mathbf{p}_1,s_1)}\Big]. \qquad (3.24)\end{aligned}$$

n.b. functional derivatives can be done in any order as arguments are all dummies (and fields are indistinguishable). After doing the next derivative we obtain

$$\begin{aligned}\frac{\delta^n w^-_{\alpha_1}(\mathbf{k}_1,t)\dots w^-_{\alpha_n}(\mathbf{k}_n,t)}{\delta w^-_{\sigma_1}(\mathbf{p}_1,s_1)\dots\delta w^-_{\sigma_n}(\mathbf{p}_n,s_n)} &= \frac{\delta^{n-2}}{\delta w^-_{\sigma_3}(\mathbf{p}_3,s_3)\dots\delta w^-_{\sigma_n}(\mathbf{p}_n,s_n)}\times\\ &\times\Big[\delta_{\alpha_1\sigma_1}\delta(\mathbf{k_1}-\mathbf{p_1})\delta(t-s_1)\frac{\delta w^-_{\alpha_2}(\mathbf{k}_2,t)\dots w^-_{\alpha_n}(\mathbf{k}_n,t)}{\delta w^-_{\sigma_2}(\mathbf{p}_2,s_2)}+\\ &+\delta_{\alpha_1\sigma_2}\delta(\mathbf{k_1}-\mathbf{p_2})\delta(t-s_2)\frac{\delta w^-_{\alpha_2}(\mathbf{k}_2,t)\dots w^-_{\alpha_n}(\mathbf{k}_n,t)}{\delta w^-_{\sigma_1}(\mathbf{p}_1,s_1)}\\ &+w^-_{\alpha_1}(\mathbf{k}_1,t)\,\frac{\delta^2 w^-_{\alpha_2}(\mathbf{k}_2,t)\dots w^-_{\alpha_n}(\mathbf{k}_n,t)}{\delta w^-_{\sigma_1}(\mathbf{p}_1,s_1)\,\delta w^-_{\sigma_2}(\mathbf{p}_2,s_2)}\Big], \qquad (3.25)\end{aligned}$$

and after all n derivatives have been taken inside we get

$$\begin{aligned}\frac{\delta^n w^-_{\alpha_1}(\mathbf{k}_1,t)\dots w^-_{\alpha_n}(\mathbf{k}_n,t)}{\delta w^-_{\sigma_1}(\mathbf{p}_1,s_1)\dots\delta w^-_{\sigma_n}(\mathbf{p}_n,s_n)} &= \Big[\Big(\,\delta_{\alpha_1\sigma_1}\delta(\mathbf{k_1}-\mathbf{p_1})\delta(t-s_1)\;\times\\ &\times\frac{\delta^{n-1}w^-_{\alpha_2}(\mathbf{k}_2,t)\dots w^-_{\alpha_n}(\mathbf{k}_n,t)}{\delta w^-_{\sigma_2}(\mathbf{p}_2,s_2)\dots\delta w^-_{\sigma_n}(\mathbf{p}_n,s_n)}\Big)+\\ &+\Big(\,\delta_{\alpha_1\sigma_2}\delta(\mathbf{k_1}-\mathbf{p_2})\delta(t-s_2)\;\times\\ &\times\frac{\delta^{n-1}w^-_{\alpha_2}(\mathbf{k}_2,t)\dots w^-_{\alpha_n}(\mathbf{k}_n,t)}{\delta w^-_{\sigma_1}(\mathbf{p}_1,s_1)\,\delta w^-_{\sigma_3}(\mathbf{p}_3,s_3)\dots\delta w^-_{\sigma_n}(\mathbf{p}_n,s_n)}\Big)+\\ &+\vdots\\ &+\Big(\,\delta_{\alpha_1\sigma_n}\delta(\mathbf{k_1}-\mathbf{p_n})\delta(t-s_n)\;\times\\ &\times\frac{\delta^{n-1}w^-_{\alpha_2}(\mathbf{k}_2,t)\dots w^-_{\alpha_n}(\mathbf{k}_n,t)}{\delta w^-_{\sigma_1}(\mathbf{p}_1,s_1)\dots\delta w^-_{\sigma_{n-1}}(\mathbf{p}_{n-1},s_{n-1})}\Big)+\\ &+\,w^-_{\alpha_1}(\mathbf{k}_1,t)\,\frac{\delta^n w^-_{\alpha_2}(\mathbf{k}_2,t)\dots w^-_{\alpha_n}(\mathbf{k}_n,t)}{\delta w^-_{\sigma_1}(\mathbf{p}_1,s_1)\dots\delta w^-_{\sigma_n}(\mathbf{p}_n,s_n)}\Big], \qquad (3.26)\end{aligned}$$

where the last term on the RHS will be equal to zero by noticing that there

is one more functional differential operator than there are elements being differentiated (remembering that each element w will give a delta function when differentiated and thus will only be differentiated once). Substituting (3.26) into (3.23), integrating out the delta functions and renaming dummy variables appropriately we have

$$\begin{aligned}\widehat{\mathcal{P}}_c^- \left\{u_{\alpha_1}^- (\mathbf{k}_1,t)\dots u_{\alpha_n}^- (\mathbf{k}_n,t)\right\} &= n u_{\alpha_1}^- (\mathbf{k}_1,t)\frac{1}{n!}\prod_{i=1}^{n-1}\int d^3p_i \int ds_i u_{\sigma_i}^- (\mathbf{p}_i,s_i)\times \\ &\times \int \mathcal{D}w_\lambda (\mathbf{z},\tau)\, P\left[w_\lambda (\mathbf{z},\tau)\right]\times \\ &\times \frac{\delta^{n-1} w_{\alpha_2}^- (\mathbf{k}_2,t)\dots w_{\alpha_n}^- (\mathbf{k}_n,t)}{\delta w_{\sigma_1}^- (\mathbf{p}_1,s_1)\dots \delta w_{\sigma_{n-1}}^- (\mathbf{p}_{n-1},s_{n-1})}. \end{aligned} \quad (3.27)$$

Canceling the n from the factorial in the denominator, we are left with the most important result

$$\widehat{\mathcal{P}}_c^- \left\{u_{\alpha_1}^- (\mathbf{k}_1,t)\dots u_{\alpha_n}^- (\mathbf{k}_n,t)\right\} = u_{\alpha_1}^- (\mathbf{k}_1,t)\,\widehat{\mathcal{P}}_c^- \left\{u_{\alpha_2}^- (\mathbf{k}_2,t)\dots u_{\alpha_n}^- (\mathbf{k}_n,t)\right\}. \quad (3.28)$$

Equations (3.19) and (3.20) are obtained by repeated application of (3.28).

The proof for (3.21) requires an extra step of expanding the $u^{+\prime}s$ in terms of $u^{-\prime}s$ as we know that the Navier-Stokes equation introduces this interdependence of $u^{-\prime}s$ on $u^{+\prime}s$ and vice versa. We can illustrate this by introducing the operator $\widehat{\mathcal{F}}_{NSE}$, which has the effect that

$$\begin{aligned}\widehat{\mathcal{F}}_{NSE} f[u^-] &= g[u^+], \\ \widehat{\mathcal{F}}_{NSE} f[u^+] &= g[u^-], \end{aligned} \quad (3.29)$$

and is simply an operational way of saying that we can use the NSE to iteratively create a power series expression for u^- in terms of u^+ and vice versa such that we can turn any expression which is only a functional of u^- into a *different* functional of u^+ only (and vice versa).

If we assume that the u^+'s are independent of the u^-'s[4] then we get the result that the conditional average turns into a normal full ensemble average

$$\widehat{\mathcal{P}}_c^- \left\{u_\lambda^+ (\mathbf{m},t)\dots u_\delta^+ (\mathbf{p},t)\right\} = \left\langle u_\lambda^+ (\mathbf{m},t)\dots u_\delta^+ (\mathbf{p},t)\right\rangle, \quad (3.30)$$

[4]note that this assumption is an extreme case which is only true for zero coupling.

similar to the *Asymptotic freedom* approximation of McComb *et. al* [64]. This result becomes clear if we set $N = 0$ in (3.22) . To see this in the functional calculations it helps to know that the assumption of the u^+ independence from u^- implies that

$$\frac{\delta u_\alpha^+(\mathbf{k},t)}{\delta u_\lambda^-(\mathbf{p},\tau)} = 0. \tag{3.31}$$

If one requires conditional averages of the form

$$\widehat{\mathcal{P}}_c^- \left\{ h\left(u_\lambda^-\left(\mathbf{m},t\right)\right) q\left(u_\delta^+\left(\mathbf{p},t\right)\right)\right\} \tag{3.32}$$

where h and q are well-behaved functions, then one needs to first expand both functions out in a power series of the argument and then expand the u^+'s dependence on the u^-'s, after which you apply the above functional form of the conditional average as normal.

3.4 Application to the Navier-Stokes equation

We will now conditionally average the incompressible Navier-Stokes equation (NSE) in spectral space

$$\left[\frac{\partial}{\partial t} + \nu_0 k^2\right] u_\alpha(\mathbf{k},t) = M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3 j u_\beta(\mathbf{j},t) u_\gamma(\mathbf{k}-\mathbf{j},t). \tag{3.33}$$

We will start by looking at the commutativity of the conditional average with the time derivative of the NSE. By studying the various integration and functional derivatives of (3.22), we can see that the time derivative commutes with the

conditional average operation because

$$\begin{aligned}
\widehat{\mathcal{P}}_c^- \left\{ \frac{\partial}{\partial t} u_\alpha(\mathbf{k},t) \right\} &= \frac{1}{N!} \prod_{i=1}^{N} \int d^3p_i \int ds_i u_{\sigma_i}^- (\mathbf{p}_i, s_i) \int \mathcal{D}w_\lambda (\mathbf{z},\tau) \times \\
&\quad \times P\left[w_\lambda (\mathbf{z},\tau)\right] \frac{\delta^N \left(\partial w_\alpha (\mathbf{k},t)/\partial t\right)}{\delta w_{\sigma_1}^- (\mathbf{p}_1, s_1) \ldots \delta w_{\sigma_N}^- (\mathbf{p}_N, s_N)} \\
&= \frac{\partial}{\partial t} \frac{1}{N!} \prod_{i=1}^{N} \int d^3p_i \int ds_i u_{\sigma_i}^- (\mathbf{p}_i, s_i) \int \mathcal{D}w_\lambda (\mathbf{z},\tau) \times \\
&\quad \times P\left[w_\lambda (\mathbf{z},\tau)\right] \frac{\delta^N w_\alpha (\mathbf{k},t)}{\delta w_{\sigma_1}^- (\mathbf{p}_1, s_1) \ldots \delta w_{\sigma_N}^- (\mathbf{p}_N, s_N)} \\
&= \frac{\partial}{\partial t} \widehat{\mathcal{P}}_c^- \left\{ u_\alpha(\mathbf{k},t) \right\}. \qquad (3.34)
\end{aligned}$$

Thus the conditional average of (3.33) results in

$$\left[\frac{\partial}{\partial t} + \nu_0 k^2 \right] \widehat{\mathcal{P}}_c^- \left\{ u_\alpha(\mathbf{k},t) \right\} = M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j \widehat{\mathcal{P}}_c^- \left\{ u_\beta(\mathbf{j},t) u_\gamma(\mathbf{k}-\mathbf{j},t) \right\}, \qquad (3.35)$$

which if we partition into high and low-wavenumber modes (see previous chapter) we get

$$\begin{aligned}
\left[\frac{\partial}{\partial t} + \nu_0 k^2 \right] u_\alpha^-(\mathbf{k},t) &= M_{\alpha\beta\gamma}^-(\mathbf{k}) \int d^3j \left[u_\beta^-(\mathbf{j},t) u_\gamma^-(\mathbf{k}-\mathbf{j},t) \right. \\
&\quad + 2u_\beta^-(\mathbf{j},t) \widehat{\mathcal{P}}_c^- \left\{ u_\gamma^+(\mathbf{k}-\mathbf{j},t) \right\} \\
&\quad \left. + \widehat{\mathcal{P}}_c^- \left\{ u_\beta^+(\mathbf{j},t) u_\gamma^+(\mathbf{k}-\mathbf{j},t) \right\} \right], \qquad (3.36)
\end{aligned}$$

and

$$\begin{aligned}
\left[\frac{\partial}{\partial t} + \nu_0 k^2 \right] \widehat{\mathcal{P}}_c^- \left\{ u_\alpha^+(\mathbf{k},t) \right\} &= M_{\alpha\beta\gamma}^+(\mathbf{k}) \int d^3j \left[u_\beta^-(\mathbf{j},t) u_\gamma^-(\mathbf{k}-\mathbf{j},t) \right. \\
&\quad + 2u_\beta^-(\mathbf{j},t) \widehat{\mathcal{P}}_c^- \left\{ u_\gamma^+(\mathbf{k}-\mathbf{j},t) \right\} \\
&\quad \left. + \widehat{\mathcal{P}}_c^- \left\{ u_\beta^+(\mathbf{j},t) u_\gamma^+(\mathbf{k}-\mathbf{j},t) \right\} \right], \qquad (3.37)
\end{aligned}$$

where we have not yet completed the conditional average of the terms involving a u^+ because we need to have an expression for them to some order in u^-. The reason for this being that the functional representation of the conditional average requires expressions in terms of u^- to operate on, otherwise it assumes that u^+ is independent of u^-. We will illustrate this further by analysing the cross term to 1st order in $M_{\alpha\beta\gamma}(\mathbf{k})$ projectors, the Reynolds term to 2nd

order and then evaluating the conditional average, as in the two-field theory of McComb *et. al.* [62], where the justification of the truncation at these orders is given.

3.4.1 Cross term

We are analysing this to first order in $M_{\alpha\beta\gamma}(\mathbf{k})$ projectors so we leave this expression as it is, and thus there is only a u^+ in the conditional average. This means that the conditional average becomes a normal ensemble average because it sees no u^- as it's argument. Instead it sees a $(u^-)^0 = 1$ and thus in the expression (3.22) for the CA we get $N = 0$ and thus

$$\begin{aligned} 2u^-_\beta(\mathbf{j},t)\widehat{\mathcal{P}}^-_c\left\{u^+_\gamma(\mathbf{k}-\mathbf{j},t)\right\} &= 2u^-_\beta(\mathbf{j},t)\frac{1}{0!}\prod_{i=1}^{0}\int d^3p_i\int ds_i u^-_{\sigma_i}(\mathbf{p}_i,s_i)\times \\ &\times\int \mathcal{D}w_\lambda(\mathbf{z},\tau)\,P\left[w_\lambda(\mathbf{z},\tau)\right]\frac{\delta^0 f\left[w^+_\alpha(\mathbf{k},t)\right]}{\delta w^-_{\sigma_1}(\mathbf{p}_1,s_1)\ldots\delta w^-_{\sigma_0}(\mathbf{p}_0,s_0)} \\ &= 2u^-_\beta(\mathbf{j},t)\int \mathcal{D}w_\lambda(\mathbf{z},\tau)\,P\left[w_\lambda(\mathbf{z},\tau)\right]w^+_\gamma(\mathbf{k}-\mathbf{j},t) \\ &= 2u^-_\beta(\mathbf{j},t)\left\langle u^+_\gamma(\mathbf{k}-\mathbf{j},t)\right\rangle \\ &= 0, \end{aligned} \tag{3.38}$$

due to the **mean** fluctuating velocity being defined as zero.

3.4.2 Reynolds term

We are to evaluate this term to 2nd order in $M_{\alpha\beta\gamma}(\mathbf{k})$ projectors, as reasoned in [62], so we will need an expression for $M^-_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\widehat{\mathcal{P}}^-_c\left\{u^+_\beta(\mathbf{j},t)u^+_\gamma(\mathbf{k}-\mathbf{j},t)\right\}$ to substitute into (3.36) . This expression is readily obtained (see next chapter)

$$\begin{aligned} M^-_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\widehat{\mathcal{P}}^-_c\left\{u^+_\beta(\mathbf{j},t)u^+_\gamma(\mathbf{k}-\mathbf{j},t)\right\} &= \int d^3j\frac{2M^-_{\alpha\beta\gamma}(\mathbf{k})M^+_{\beta\delta\epsilon}(\mathbf{j})}{\nu_0 j^2+\nu_0\left|\mathbf{k}-\mathbf{j}\right|^2}\times \\ &\times\int d^3p\Big\{\widehat{\mathcal{P}}^-_c\left\{u^-_\delta(p,t)u^-_\epsilon(\mathbf{j}-\mathbf{p},t)u^+_\gamma(\mathbf{k}-\mathbf{j},t)\right\} \\ &\quad+2\widehat{\mathcal{P}}^-_c\left\{u^-_\delta(p,t)u^+_\epsilon(\mathbf{j}-\mathbf{p},t)u^+_\gamma(\mathbf{k}-\mathbf{j},t)\right\} \\ &\quad+\widehat{\mathcal{P}}^-_c\left\{u^+_\delta(p,t)u^+_\epsilon(\mathbf{j}-\mathbf{p},t)u^+_\gamma(\mathbf{k}-\mathbf{j},t)\right\}\Big\}, \end{aligned} \tag{3.39}$$

to 2nd order in $M_{\alpha\beta\gamma}(\mathbf{k})$ projectors as required. Evaluating this conditional average we will see that the first term on the RHS will be zero as it involves a conditional average over one u^+ which the conditional average sees as asymptotically free in a similar way to the evaluation of the cross term. We deal with the last term by realising that we will need a further expression for this such that it involves terms linear in u^- to facilitate the RG calculation. However, this will introduce terms higher order in $M_{\alpha\beta\gamma}(\mathbf{k})$ projectors, so this term is neglected to 2nd order. This leaves us with

$$\begin{aligned} M^-_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j \widehat{\mathcal{P}}^-_c \left\{ u^+_\beta(\mathbf{j},t) u^+_\gamma(\mathbf{k}-\mathbf{j},t) \right\} &= \int d^3j \frac{2M^-_{\alpha\beta\gamma}(\mathbf{k}) M^+_{\beta\delta\epsilon}(\mathbf{j})}{\nu_0 j^2 + \nu_0 \left|\mathbf{k}-\mathbf{j}\right|^2} \times \\ &\times \int d^3p 2u^-_\delta(p,t) \widehat{\mathcal{P}}^-_c \left\{ u^+_\epsilon(\mathbf{j}-\mathbf{p},t) u^+_\gamma(\mathbf{k}-\mathbf{j},t) \right\} \\ &= \int d^3j \frac{2M^-_{\alpha\beta\gamma}(\mathbf{k}) M^+_{\beta\delta\epsilon}(\mathbf{j})}{\nu_0 j^2 + \nu_0 \left|\mathbf{k}-\mathbf{j}\right|^2} \times \\ &\times \int d^3p 2u^-_\delta(p,t) \left\langle u^+_\epsilon(\mathbf{j}-\mathbf{p},t) u^+_\gamma(\mathbf{k}-\mathbf{j},t) \right\rangle, \quad (3.40) \end{aligned}$$

where in the last line we have assumed asymptotic freedom as earlier, because we are working to 2nd order. This leaves us with (3.36) looking like

$$\begin{aligned} \left[\frac{\partial}{\partial t} + \nu_0 k^2 \right] u^-_\alpha(\mathbf{k},t) &= M^-_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j u^-_\beta(\mathbf{j},t) u^-_\gamma(\mathbf{k}-\mathbf{j},t) \\ &+ \left\{ \int d^3j \frac{2M^-_{\alpha\beta\gamma}(\mathbf{k}) M^+_{\beta\delta\epsilon}(\mathbf{j})}{\nu_0 j^2 + \nu_0 \left|\mathbf{k}-\mathbf{j}\right|^2} \times \right. \\ &\times \left. \int d^3p 2u^-_\delta(p,t) \left\langle u^+_\epsilon(\mathbf{j}-\mathbf{p},t) u^+_\gamma(\mathbf{k}-\mathbf{j},t) \right\rangle \right\}, \quad (3.41) \end{aligned}$$

where the evaluation of the last term results in the increment to the viscosity in the two-field theory RG of McComb *et. al.*

3.4.3 Important notes on the NSE and conditional averaging

Due to the deterministic nature of the velocity fluctuations i.e. they are governed by the NSE, one can theoretically write an expression for the u^+ completely in terms of the u^- using $\widehat{\mathcal{F}}_{NSE}$ introduced earlier. The NSE provides a moment hierarchy which will do exactly this such that we result with an expression for u^+ in terms of an infinite expansion of u^-. This implies that if we then

go and take a conditional average of this expression it will be unaffected and thus we obtain the result

$$\widehat{\mathcal{P}}_c^- \left\{u^+\right\} = u^+. \tag{3.42}$$

So when we expand the u^+ in terms of u^- and then truncate to some order, we are in effect approximating an expression for u^+. This is a very important point to remember, that the conditional average, as mentioned in section 3.2, is simply a way of approximating u^+.

3.5 Derivation of the conditional average

To simplify the derivation we will use the example of a functional $g[u^-(k)]$ of a scalar function $u^-(k)$ that has been filtered in the low wavenumber domain such that its support is $]0, k_c[$, where k_c is some wavenumber partitioning the low and high wavenumbers. The ensemble average of $g[u^-(k)]$ is

$$\left\langle g[u^-(k)]\right\rangle = \int \mathcal{D}w(z) P\left[w(z)\right] g[w^-(k)]. \tag{3.43}$$

We start by writing the low and high wavenumber parts separately

$$\left\langle g[u^-(k)]\right\rangle = \int \mathcal{D}w^-(z)\mathcal{D}w^+(z) P\left[w^-(z), w^+(z)\right] g[w^-(k)], \tag{3.44}$$

to facilitate the following discussion. We then assume that we can separate P into a product of two *different* distributions P_- and P_+

$$\left\langle g[u^-(k)]\right\rangle = \int \mathcal{D}w^-(z)\mathcal{D}w^+(z) P_-\left[w^-(z)\right] P_+\left[w^+(z)\right] g[w^-(k)]\ . \tag{3.45}$$

This step can be justified by the assumption that short wavelength modes should decouple in the average from the high wavelength modes. However, one must still remember that the NSE is in principle deterministic as illustrated in schematic Figure 3.2b.

We will now need the following theorem (see Appendix A) which is the Taylor series generalisation to functionals. This states that any functional $G\left[x(s)\right]$ continuous in the continuous functions $x(s)$ over $[a, b]$ may be represented as

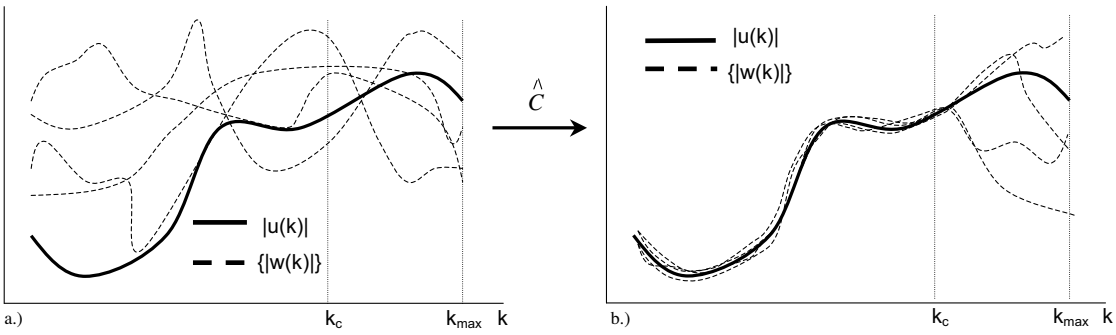

Figure 3.2: Schematic showing realisations $w(k)$ around chosen realisation $u(k)$ a.) before applying the conditional operator $\hat{\mathcal{C}}$ and b.) after application of $\hat{\mathcal{C}}$.

an expansion around a desired function/path $y(s)$ by

$$\begin{aligned} G\left[x(s)\right] &= \chi_0(s) + \int_a^b \chi_1(t_1)\left(x(t_1) - y(t_1)\right) dt_1 + \\ &+ \int_a^b \int_a^b \chi_2(t_1,t_2)\left(x(t_1) - y(t_1)\right)\left(x(t_2) - y(t_2)\right) dt_1 dt_2 + \\ &+ \cdots + \\ &+ \left[\int_a^b \cdots \int_a^b dt_1 \cdots dt_n \chi_n(t_1,\ldots,t_n) \times \right. \\ &\times \left. \left(x(t_1) - y(t_1)\right) \cdots \left(x(t_n) - y(t_n)\right)\right], \end{aligned} \tag{3.46}$$

in the limit $n \to \infty$, where the χ's are the susceptibilities given by

$$\chi_n(t_1,\ldots,t_n) = \frac{1}{n!} \left.\frac{\delta^{(n)} G[x(s)]}{\delta x(t_1)\delta x(t_2)\cdots\delta x(t_n)}\right|_{x=y}. \tag{3.47}$$

Now, expanding $g[w^-(k)]$ from (3.45) in a functional Taylor series around a specific realisation $u^-(k) \in \{w^-(k)\}$ we get

$$\begin{aligned} g[w^-(k)] &= g[u^-(k)] + \\ &+ \int d\zeta \left(w^-(\zeta) - u^-(\zeta)\right) \left.\frac{\delta g[w^-(k)]}{\delta w^-(\zeta)}\right|_{w^-(\zeta)=u^-(\zeta)} \\ &+ \mathcal{O}\left(\left(w^- - u^-\right)^2\right) \end{aligned} \tag{3.48}$$

A schematic of the realisations or paths $\{w(k)\}$ with respect to the specific realisation we are expanding around, is shown in Figure. 3.2a.

If we substitute (3.48) in (3.45) we obtain

$$\begin{aligned}\left\langle g[u^-(k)]\right\rangle &= \int \mathcal{D}w^-(z)\mathcal{D}w^+(z)P_-\left[w^-(z)\right]P_+\left[w^+(z)\right]g[u^-(k)]+\\ &+\left\{\int \mathcal{D}w^-(z)\mathcal{D}w^+(z)P_-\left[w^-(z)\right]P_+\left[w^+(z)\right]\times\right.\\ &\left.\times\int d\zeta\left(w^-(\zeta)-u^-(\zeta)\right)\left.\frac{\delta g[w^-(k)]}{\delta w^-(\zeta)}\right|_{w^-(\zeta)=u^-(\zeta)}\right\}\\ &+\left\langle\mathcal{O}\left(\left(w^--u^-\right)^2\right)\right\rangle. \qquad (3.49)\end{aligned}$$

Simplifying this expression by doing the relevant averaging operations remembering that both P_- and P_+ are normalised distributions, we obtain

$$\begin{aligned}\left\langle g[u^-(k)]\right\rangle &= g[u^-(k)]+\\ &+\int d\zeta\left(\left\langle u^-(\zeta)\right\rangle-u^-(\zeta)\right)\left.\frac{\delta g[w^-(k)]}{\delta w^-(\zeta)}\right|_{w^-(\zeta)=u^-(\zeta)}+\\ &\mathcal{O}\left(\left(\left\langle u^-\right\rangle-u^-\right)^2\right). \qquad (3.50)\end{aligned}$$

Now if we compare the result in equation (3.50) with our original expression in equation (3.45) we can see that we will obtain the same result if we had used the following representation for the probability distribution functional $P_-\left[w^-(z)\right]$

$$\begin{aligned}P_-\left[w^-(z)\right]\Phi &= \delta[w^-(z)-u^-(z)]\Phi+\\ &+\int d\zeta\left(\left\langle u^-(\zeta)\right\rangle-u^-(\zeta)\right)\delta[w^-(z)-u^-(z)]\frac{\delta\Phi}{\delta w^-(\zeta)}+\\ &+higher\ order\ terms\ , \qquad (3.51)\end{aligned}$$

where Φ is placeholder for a function or functional; remembering that P_- is a generalised function and is defined by its action on a test function (see Appendix A). Rewriting equation (3.45) using (3.51) we now obtain

$$\begin{aligned}\left\langle g[u^-(k)]\right\rangle &= \int \mathcal{D}w^-(z)\mathcal{D}w^+(z)P_+\left[w^+(z)\right]\delta[w^-(z)-u^-(z)]g[w^-(k)]+\\ &+\left\{\int \mathcal{D}w^-(z)\mathcal{D}w^+(z)P_+\left[w^+(z)\right]\times\right.\\ &\left.\times\int d\zeta\overline{\Delta u^-(\zeta)}\delta[w^-(z)-u^-(z)]\frac{\delta}{\delta w^-(\zeta)}g[w^-(k)]\right\}+\\ &+\mathcal{O}\left(\left(\overline{\Delta u^-}\right)^2\right), \qquad (3.52)\end{aligned}$$

where to facilitate the next calculation we have rewritten the term $\langle u^- \rangle - u^-$, which represents fluctuations around the mean, as $\overline{\Delta u^-(\zeta)}$.

3.5.1 The conditional projection

Now we introduce the conditional projector $\hat{\mathcal{C}}$ such that when $\hat{\mathcal{C}}$ is applied to $\langle\ \rangle$, the full ensemble average, it restricts the realisations comprising the ensemble such that

$$\lim \left| \overline{\Delta u^-(\zeta)} \right| = \epsilon \to 0 \qquad \forall \zeta. \tag{3.53}$$

The realisations of the conditional ensemble can be viewed schematically as in Figure. 3.2b.

Applying $\hat{\mathcal{C}}$ to (3.52) we obtain

$$\begin{aligned} \hat{\mathcal{C}} \left\{ \left\langle g[u^-(k)] \right\rangle \right\} &= \lim_{\epsilon \to 0} \int \mathcal{D}w^-(z) \mathcal{D}w^+(z) P_+ \left[w^+(z) \right] \delta[w^-(z) - u^-(z)] g[w^-(k)] + \\ &\quad + \lim_{\epsilon \to 0} \mathcal{O}\left(\overline{\Delta u^-} \right), \end{aligned} \tag{3.54}$$

where terms in orders of $\overline{\Delta u^-}$ have now become negligible in the limit. The limit is necessary because it ensures that we have an ensemble left for averaging over the w^+'s. If we just stated that we are picking all realisations, w, which have their low-*k* parts *identical* to u^- we would automatically be fixing the w^+ to u^+ due to the deterministic nature of the NSE and thus will have no ensemble to average over. Our aim is to constrain $w^-(k)$ 'without' constraining $w^+(k)$ too much.

Thus we have as our final result that the conditional projector will give

$$\lim_{\epsilon \to 0} P_- \left[w^-(z) \right] = \delta \left[w^-(z) - u^-(z) \right]. \tag{3.55}$$

The conditional projector will also change $P_+ [w^+(z)]$; thus

$$\lim_{\epsilon \to 0} P_+ \left[w^+(z) \right] \to P_{\mathcal{C}+} \left[w^+(z) \right], \tag{3.56}$$

where $P_{\mathcal{C}+}$ indicates a change from P_+ to the conditional probability distribution functional. Hence, after integrating out the delta functional[5] in equation (3.54)

[5]This is strictly speaking a product of an infinitely many delta functions for which z takes all values from 0 to k_c.

we get to the desired behaviour

$$\begin{aligned}\hat{\mathcal{C}}\left\{g[u^-(k)]\right\} &= \int \mathcal{D}w^+(z)P_{\mathcal{C}+}\left[w^+(z)\right]g[u^-(k)] \\ &= g[u^-(k)]\int \mathcal{D}w^+(z)P_{\mathcal{C}+}\left[w^+(z)\right] \\ &= g[u^-(k)]\,, \end{aligned} \tag{3.57}$$

where the last line uses the normalisation property of the probability distribution functional.

3.5.2 Asymptotic Freedom

Now if we generalise to the average of expressions which involve u^+, taking the evaluation of $\hat{\mathcal{C}}\left\{\langle u^-(k)u^+(j)\rangle\right\}$ as an example, we will get the result that

$$\begin{aligned}\hat{\mathcal{C}}\left\{\left\langle u^-(k)u^+(j)\right\rangle\right\} &= u^-(k)\int \mathcal{D}w^+(z)P_{\mathcal{C}+}\left[w^+(z)\right]w^+(j) \\ &= u^-(k)\hat{\mathcal{C}}\left\{\left\langle u^+(j)\right\rangle\right\}. \end{aligned} \tag{3.58}$$

To evaluate what $\hat{\mathcal{C}}\left\{\langle u^+(j)\rangle\right\}$ is, we introduce the approximation of an *asymptotic freedom* operator $\hat{\mathcal{A}}$. The application of this operator causes the average in question to be performed such that it treats all entities (velocities in our case) in the argument of the average as corresponding to the wavenumber modes in the neighbourhood of k_{max} i.e. far from the partition wavenumber k_c. This can be seen by help of the schematic in Figure 3.3.

Comparing Figure 3.3 after the application of $\hat{\mathcal{A}}$ to Figure 3.2a., we see that the high wavenumber band ensemble now resembles the realisations of the *free* unrestricted ensemble. This has been previously discussed by McComb *et. al.* [62, 64], where it was known as the *hypothesis of local chaos* i.e the ability of the signal to decorrelate as it progresses through the modes k. With this in mind we may make the following approximation

$$\hat{\mathcal{A}}\left\{\hat{\mathcal{C}}\left\{\left\langle u^+(j)\right\rangle\right\}\right\} = \left\langle u^+(j)\right\rangle\ . \tag{3.59}$$

By writing relation (3.59) we are effectively holding the full ensemble as representative of the sub-ensemble *in the subgrid band only*. We must take care to remember that this is only really true at the UV cut-off k_{max}. In the next chap-

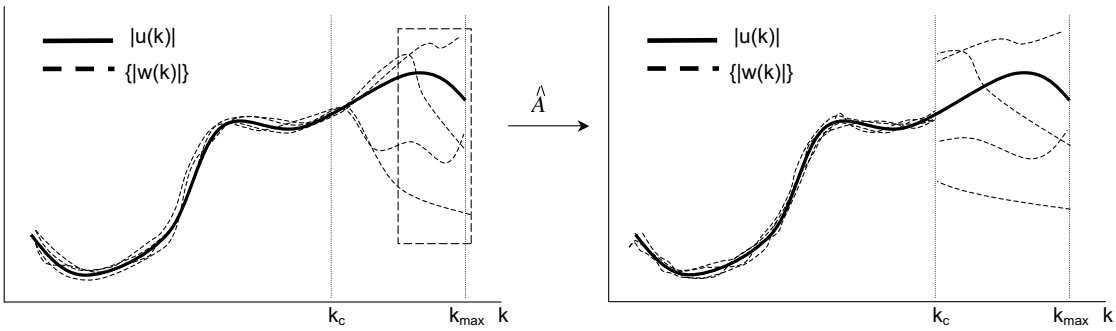

Figure 3.3: A schematic showing the effect of the asymptotic freedom operator $\hat{\mathcal{A}}$, as approximating the u^+ modes to behave like modes in the neighbourhood of k_{max}.

ter we will modify the asymptotic freedom operation, and to conform with the previous results of McComb *et. al.*, take the subgrid band to be given by a full ensemble average on a linear analytic continuation from k_{max}.

This completes our derivation of the conditional average. Essentially, the conditional average operation denoted by $\widehat{\mathcal{P}}_c^-$ which was mentioned earlier in this chapter, is just the application (in the right order) of the two operators introduced in this section

$$\widehat{\mathcal{P}}_c^- \{\ \} \equiv \hat{\mathcal{A}} \left\{ \hat{\mathcal{C}} \{\langle\ \rangle\} \right\}. \tag{3.60}$$

We are now ready to get to the form (3.22) of the conditional average that was arrived at, earlier, from construction. Again, one opts for a simpler example, that can be generalised to the more thorough and complicated form (3.22) in a systematically easy but laborious way. Thus, we shall start by looking at the ensemble average of $u^-(k)u^-(j)u^+(l)u^+(m)$ given by

$$\left\langle u^-(k)u^-(j)u^+(l)u^+(m) \right\rangle = \int_R \mathcal{D}w\,(z)\,P\,[w\,(z)]\,\underline{w^-(k)w^-(j)}w^+(l)w^+(m). \tag{3.61}$$

We now functionally Taylor expand the underlined term around some path u^-

$$\begin{aligned} w^-(k)w^-(j) &= u^-(k)u^-(j) \\ &\quad + \int dp \left(w^-(p) - u^-(p)\right) \frac{\delta}{\delta w^-(p)} \left(w^-(k)w^-(j)\right)\Big|_{w^-=u^-} \\ &\quad + \Bigg[\int\int dpds \left(w^-(p) - u^-(p)\right)\left(w^-(s) - u^-(s)\right) \times \\ &\quad \times \frac{1}{2!} \frac{\delta^2 \left(w^-(k)w^-(j)\right)}{\delta w^-(s)\delta w^-(p)}\Big|_{w^-=u^-}\Bigg] \ . \end{aligned} \tag{3.62}$$

Doing the functional differentiation and then integrating the resultant terms, we will find that many terms will cancel and that the only relevant term remaining in this expansion is

$$\int dsdp w^-(s)w^-(p)\frac{1}{2!}\frac{\delta^2 w^-(k)w^-(j)}{\delta w^-(s)\delta w^-(p)}, \tag{3.63}$$

which when evaluated results in $w^-(k)w^-(j)$, as it should to make the expansion in (3.62) consistent. This result is valid for all polynomial expressions of degree n. Note that we have not included the path condition on the differential operator as we know that it is of no use here since the functional differentiation will only yield delta functions. Substituting (3.63) in (3.61) we obtain

$$\begin{aligned} RHS\ of\ (3.61) &= \int_R \mathcal{D}w\left(z\right) P\left[w\left(z\right)\right] \int\int dsdp w^-(s)w^-(p) \times \\ &\quad \times \frac{1}{2!}\frac{\delta^2 w^-(k)w^-(j)}{\delta w^-(s)\delta w^-(p)} w^+(l)w^+(m) \ . \end{aligned} \tag{3.64}$$

Rearranging the integrals and assuming that the w^+ is independent of w^- so we may include them in the differential operators, we can write

$$\frac{1}{2!}\int\int dsdp \int_R \mathcal{D}w\left(z\right) P\left[w\left(z\right)\right] w^-(s)w^-(p)\frac{\delta^2 w^-(k)w^-(j)w^+(l)w^+(m)}{\delta w^-(s)\delta w^-(p)}. \tag{3.65}$$

Applying the conditional projection operator $\hat{\mathcal{C}}$, we obtain the result

$$\frac{1}{2!}\int\int dsdp \int_R \mathcal{D}w^+\left(z\right) P_{\mathcal{C}+}\left[w^+\left(z\right)\right] u^-(s)u^-(p)\frac{\delta^2 w^-(k)w^-(j)w^+(l)w^+(m)}{\delta w^-(s)\delta w^-(p)}. \tag{3.66}$$

We can now take the u^- through the functional integration

$$\frac{1}{2!}\int\int dsdpu^-(s)u^-(p)\int_R \mathcal{D}w^+(z)\,P_{\mathcal{C}+}\left[w^+(z)\right]\frac{\delta^2 w^-(k)w^-(j)w^+(l)w^+(m)}{\delta w^-(s)\delta w^-(p)}, \tag{3.67}$$

and apply the asymptotic freedom operator $\hat{\mathcal{A}}$ to obtain

$$\frac{1}{2!}\int\int dsdpu^-(s)u^-(p)\int_R \mathcal{D}w^+(z)\,P_{+}\left[w^+(z)\right]\frac{\delta^2 w^-(k)w^-(j)w^+(l)w^+(m)}{\delta w^-(s)\delta w^-(p)}. \tag{3.68}$$

Changing the measure of the functional integration from $\mathcal{D}w^+(z)\,P_+\left[w^+(z)\right]$ to $\mathcal{D}w(z)\,P\left[w(z)\right]$ such that we have the integration over all the path instead of the partitioned paths, does not change any of the results from (3.68). With this in mind we make the final change

$$\frac{1}{2!}\int\int dsdpu^-(s)u^-(p)\int_R \mathcal{D}w(z)\,P\left[w(z)\right]\frac{\delta^2 w^-(k)w^-(j)w^+(l)w^+(m)}{\delta w^-(s)\delta w^-(p)}, \tag{3.69}$$

and thus obtain a particular case of our earlier form (3.22) for an expression where the w^+ are assumed independent from the w^-. It is a straight-forward task to then generalise this result to the form given by (3.22).

3.6 Discussion

It is quite clear that the expression in (3.69) is not unique and one could have obtained other forms for the conditional average. For example, one did not need to change the measure in (3.68). One query out of many could be that: why does one need to expand the u^- part into a differential Taylor series? Why not leave the u^- part as it is? The answer being that firstly, this allows an operational form of the conditional average to be constructed. Secondly it can be very practical in situations where the expression being averaged involves mixed arguments; the function or functional can then be expanded into a power series of the desired function argument and (3.22) can be easily applied.

In the practical applications of the conditional average to problems, we do not need to resort to doing functional integrals each time we want to use the CA. We may simply just treat the averaging procedure like the filtered ensemble average but being careful to remember that the construction of this conditional

average is not the same, and certain crucial approximations were necessary to get to this form. This also brings us to another point, that although one might consider the analysis so far as just one big exercise to justify using the filtered ensemble average, it is in fact showing that the filtered ensemble average is just one approximation (crude as it is) of a *true* conditional average. This is the main point of the whole chapter. In fact, in later chapters we will change some of the approximations to refine the CA better. This will primarily concern the changing of the asymptotic freedom operation to only apply at the UV limit cut-off wavenumber (Chapter 4), and extending the CA to accommodate two distinguishable fields (Chapter 6).

Lastly, we should remember that the techniques used in constructing the conditional average and also the conditional average itself, need not be confined to the situation of Navier-Stokes turbulence only. The techniques are quite general and a similar analysis should still apply to many problems in many-body physics where one has strong mode-mode coupling.

Chapter 4

The Renormalization Group calculation

In Chapter 2 we discussed the application of dynamical RG to the modeling of turbulence. In this chapter we will be discussing in much more detail the 'Conditional mode elimination' procedure of McComb *et al.* in which the use of a conditional average with asymptotic freedom is employed. The previous chapter was concerned with the formal derivation of a functional operator which combines the conditional average and the asymptotic freedom limit in one convenient operation. Similar to previous work we will now implicitly assume this procedure instead of applying the functional operator at every stage that it is needed. The RG procedure given below is essentially the same as the one presented by McComb & Johnston [64] except that we will be doing the coarse-graining before the rescaling as is the usual case in RG studies.

4.1 Dimensionless NSE

To facilitate the calculation and justify the approximations made, we will find it useful to write the NSE in its dimensionless form where the wavenumbers are defined on the interval $[0, 1]$. We introduce the scaled variables

$$\hat{\mathbf{k}} = \mathbf{k}/k_0, \tag{4.1}$$

$$\hat{t} = t/\tau(k_0), \tag{4.2}$$

and

$$\hat{u}_\alpha(\hat{\mathbf{k}}, \hat{t}) = u_\alpha(\mathbf{k}, t)/V(k_0), \tag{4.3}$$

where $k_0 \equiv k_{max}$ is our maximum wavenumber (UV cut-off), $\tau(k_0)$ is as yet an undetermined time-scale parameterised at $k = k_0$, and $V(k_0)$ is the r.m.s. value of a velocity parameterised at $k = k_0$ and is calculated at any given wavenumber k by

$$V^2(k) = \frac{1}{k^3} \int d^3j \, \langle u_\alpha(\mathbf{k}, t) u_\alpha(\mathbf{j}, t) \rangle \ , \tag{4.4}$$

such that it includes the average effect of all the other velocity modes. We will also be assuming that the viscosity is in general a function of wavenumber k; even though in the first shell removal of the RG process it is constant, for later shell removals it will not be. This simply means making the replacement $\nu_0 \to \nu_0(k)$ in the NSE. Substituting the above scaled variables into the NSE results in the dimensionless NSE

$$\left(\frac{\partial}{\partial \hat{t}} + \hat{\nu}_0(\hat{k}) \hat{k}^2 \right) \hat{u}_\alpha(\hat{\mathbf{k}}, \hat{t}) = R_0(k_0) M_{\alpha\beta\gamma}(\hat{\mathbf{k}}) \int d^3\hat{j} \hat{u}_\beta(\hat{\mathbf{j}}, \hat{t}) \hat{u}_\gamma(\hat{\mathbf{k}} - \hat{\mathbf{j}}, \hat{t}) + \hat{f}_\alpha(\hat{\mathbf{k}}, \hat{t}), \tag{4.5}$$

where the dimensionless viscosity is given by

$$\begin{aligned} \hat{\nu}_0(\hat{k}) &= \tau(k_0) k_0^2 \nu_0(k) \\ &= \tau(k_0) k_0^2 \nu_0(k_0 \hat{k}), \end{aligned} \tag{4.6}$$

and the dimensionless forcing is

$$\hat{f}_\alpha(\hat{\mathbf{k}}, \hat{t}) = f_\alpha(\mathbf{k}, t) \tau(k_0)/V(k_0), \tag{4.7}$$

and

$$R_0(k_0) = \tau(k_0) V(k_0) k_0^4 \tag{4.8}$$

is the (dimensionless) *local* Reynolds number; local in the sense that it is parameterised on a wavenumber which in our case is k_0 [68]. We will be using the fact that $R_0(k_0)$ is small ($R_0(k_0) < 1$) in the UV range as detailed in Chapter

2, to justify our approximations. To save space we will not write the forcing in the NSE but will assume its presence implicitly. For the purposes of our RG calculation which is concerned with the intermediate and large wavenumbers, this will have no effect as we will assume that similar to real lab flows, the forcing is only acting on large scales i.e. small wavenumbers. The forcing is there simply to sustain the turbulence and, in our case, to keep it stationary i.e. the statistics of the turbulence to remain time independent.

4.2 Conditional mode elimination

We will now begin to implement the RG procedure on the dimensionless NSE (4.5) by starting with the removal of the first shell.

As described in Chapter 2, the first stage of the RG procedure is the coarse-graining. In order to do this we introduce a partitioning parameter h, such that $k_n = h^n k_0$, and the width, η, of the shell being coarse-grained is given by $\eta = 1 - h$. We start by partitioning equation (4.5) into low-k wavenumber modes

$$\left(\frac{\partial}{\partial \hat{t}} + \hat{\nu}_0(\hat{k})\hat{k}^2\right)\hat{u}_\alpha^-(\hat{\mathbf{k}},\hat{t}) = R_0 M^-_{\alpha\beta\gamma}(\hat{\mathbf{k}})\int d^3\hat{j}\left\{\hat{u}_\beta^-(\hat{\mathbf{j}},\hat{t})\hat{u}_\gamma^-(\hat{\mathbf{k}}-\hat{\mathbf{j}},\hat{t})\right.$$
$$\left. 2\hat{u}_\beta^-(\hat{\mathbf{j}},\hat{t})\hat{u}_\gamma^+(\hat{\mathbf{k}}-\hat{\mathbf{j}},\hat{t}) + \hat{u}_\beta^+(\hat{\mathbf{j}},\hat{t})\hat{u}_\gamma^+(\hat{\mathbf{k}}-\hat{\mathbf{j}},\hat{t})\right\}, \quad (4.9)$$

defined over $0 \leq \hat{k} \leq h$, and high-k wavenumber modes

$$\left(\frac{\partial}{\partial \hat{t}} + \hat{\nu}_0(\hat{k})\hat{k}^2\right)\hat{u}_\alpha^+(\hat{\mathbf{k}},\hat{t}) = R_0 M^+_{\alpha\beta\gamma}(\hat{\mathbf{k}})\int d^3\hat{j}\left\{\hat{u}_\beta^-(\hat{\mathbf{j}},\hat{t})\hat{u}_\gamma^-(\hat{\mathbf{k}}-\hat{\mathbf{j}},\hat{t})\right.$$
$$\left. 2\hat{u}_\beta^-(\hat{\mathbf{j}},\hat{t})\hat{u}_\gamma^+(\hat{\mathbf{k}}-\hat{\mathbf{j}},\hat{t}) + \hat{u}_\beta^+(\hat{\mathbf{j}},\hat{t})\hat{u}_\gamma^+(\hat{\mathbf{k}}-\hat{\mathbf{j}},\hat{t})\right\}, \quad (4.10)$$

defined over $h \leq \hat{k} \leq 1$; where we have also shortened the notation for the local Reynolds number $R_0 \equiv R_0(k_0)$. The next step is to average out the effects of the high-k modes in equation (4.9). We do this by applying the conditional average (CA) operation we formulated in the previous chapter, to equation (4.9)

$$\left(\frac{\partial}{\partial \hat{t}} + \hat{\nu}_0(\hat{k})\hat{k}^2\right)\hat{u}_\alpha^-(\hat{\mathbf{k}},\hat{t}) = R_0 M^-_{\alpha\beta\gamma}(\hat{\mathbf{k}})\int d^3\hat{j}\left\{\hat{u}_\beta^-(\hat{\mathbf{j}},\hat{t})\hat{u}_\gamma^-(\hat{\mathbf{k}}-\hat{\mathbf{j}},\hat{t})\right.$$
$$\left. +2\hat{\mathcal{P}_C}\left\{\hat{u}_\beta^-(\hat{\mathbf{j}},\hat{t})\hat{u}_\gamma^+(\hat{\mathbf{k}}-\hat{\mathbf{j}},\hat{t})\right\} + \hat{\mathcal{P}_C}\left\{\hat{u}_\beta^+(\hat{\mathbf{j}},\hat{t})\hat{u}_\gamma^+(\hat{\mathbf{k}}-\hat{\mathbf{j}},\hat{t})\right\}\right\}, \quad (4.11)$$

where the CA of the LHS and the first term of the RHS have been evaluated. To obtain form invariance we need to evaluate the last two terms on the RHS of (4.11) and write them in a way such that they act like terms linear in $\hat{u}^-_\alpha(\hat{\mathbf{k}},\hat{t})$. We will refer to these terms as the 'cross' (u^-u^+) and 'Reynolds' (u^+u^+) terms respectively. To evaluate the cross-term we notice that it is already in the form where it is linear in u^-. This seems to imply that we can evaluate the approximation of the CA of this term immediately. We will return to this assumption in the next chapter. This being so, when the CA of the cross term is evaluated we see that

$$\begin{aligned}\hat{\mathcal{P}}_{\mathcal{C}}\left\{\hat{u}^-_\beta(\hat{\mathbf{j}},\hat{t})\hat{u}^+_\gamma(\hat{\mathbf{k}}-\hat{\mathbf{j}},\hat{t})\right\} &= \hat{u}^-_\beta(\hat{\mathbf{j}},\hat{t})\hat{\mathcal{P}}_{\mathcal{C}}\left\{\hat{u}^+_\gamma(\hat{\mathbf{k}}-\hat{\mathbf{j}},\hat{t})\right\}\\ &= 0, \end{aligned} \tag{4.12}$$

as evaluated in the previous chapter. This leaves us with evaluating the CA of the Reynolds term which we need to get into a form which is linear in u^-. We do this by constructing an equation for $\hat{u}^+_\beta(\hat{\mathbf{j}},\hat{t})$ through the NSE, and then multiplying this by $\hat{u}^+_\gamma(\hat{\mathbf{k}}-\hat{\mathbf{j}},\hat{t})$. We then construct an equation for $\hat{u}^+_\gamma(\hat{\mathbf{k}}-\hat{\mathbf{j}},\hat{t})$ and multiply this with $\hat{u}^+_\beta(\hat{\mathbf{j}},\hat{t})$. Finally by adding the two resulting equations, some manipulation of dummy indices and variables, and applying the CA we obtain the result

$$\begin{aligned}&\left(\frac{\partial}{\partial\hat{t}}+\hat{\nu}_0(\hat{j})\hat{j}^2+\hat{\nu}_0(|\hat{\mathbf{k}}-\hat{\mathbf{j}}|)|\hat{\mathbf{k}}-\hat{\mathbf{j}}|^2\right)\hat{\mathcal{P}}_{\mathcal{C}}\left\{\hat{u}^+_\beta(\hat{\mathbf{j}},\hat{t})\hat{u}^+_\gamma(\hat{\mathbf{k}}-\hat{\mathbf{j}},\hat{t})\right\}=\\ &\quad=2R_0M^+_{\beta\delta\epsilon}(\hat{\mathbf{j}})\int d^3\hat{p}\left[\hat{\mathcal{P}}_{\mathcal{C}}\left\{\hat{u}^-_\delta(\hat{\mathbf{p}},\hat{t})\hat{u}^-_\epsilon(\hat{\mathbf{j}}-\hat{\mathbf{p}},\hat{t})\hat{u}^+_\gamma(\hat{\mathbf{k}}-\hat{\mathbf{j}},\hat{t})\right\}\right.\\ &\qquad+2\hat{\mathcal{P}}_{\mathcal{C}}\left\{\hat{u}^-_\delta(\hat{\mathbf{p}},\hat{t})\hat{u}^+_\epsilon(\hat{\mathbf{j}}-\hat{\mathbf{p}},\hat{t})\hat{u}^+_\gamma(\hat{\mathbf{k}}-\hat{\mathbf{j}},\hat{t})\right\}\\ &\qquad\left.+\hat{\mathcal{P}}_{\mathcal{C}}\left\{\hat{u}^+_\delta(\hat{\mathbf{p}},\hat{t})\hat{u}^+_\epsilon(\hat{\mathbf{j}}-\hat{\mathbf{p}},\hat{t})\hat{u}^+_\gamma(\hat{\mathbf{k}}-\hat{\mathbf{j}},\hat{t})\right\}\right]\quad. \end{aligned} \tag{4.13}$$

Inverting the linear operator on the LHS of equation (4.13) by means of an integrating factor we obtain

$$\begin{aligned}\hat{\mathcal{P}}_{\mathcal{C}}\left\{\hat{u}^+_\beta(\hat{\mathbf{j}},\hat{t})\hat{u}^+_\gamma(\hat{\mathbf{k}}-\hat{\mathbf{j}},\hat{t})\right\} &= R_0\int_{-\infty}^{\hat{t}}d\hat{s}e^{-(\hat{\nu}_0(\hat{j})\hat{j}^2+\hat{\nu}_0(|\hat{\mathbf{k}}-\hat{\mathbf{j}}|)|\hat{\mathbf{k}}-\hat{\mathbf{j}}|^2)(\hat{t}-\hat{s})}2M^+_{\beta\delta\epsilon}(\hat{\mathbf{j}})\times\\ &\quad\times\int d^3\hat{p}\left[\hat{\mathcal{P}}_{\mathcal{C}}\left\{\hat{u}^-_\delta(\hat{\mathbf{p}},\hat{s})\hat{u}^-_\epsilon(\hat{\mathbf{j}}-\hat{\mathbf{p}},\hat{s})\hat{u}^+_\gamma(\hat{\mathbf{k}}-\hat{\mathbf{j}},\hat{s})\right\}\right.\\ &\quad+2\hat{\mathcal{P}}_{\mathcal{C}}\left\{\hat{u}^-_\delta(\hat{\mathbf{p}},\hat{s})\hat{u}^+_\epsilon(\hat{\mathbf{j}}-\hat{\mathbf{p}},\hat{s})\hat{u}^+_\gamma(\hat{\mathbf{k}}-\hat{\mathbf{j}},\hat{s})\right\}\\ &\quad\left.+\hat{\mathcal{P}}_{\mathcal{C}}\left\{\hat{u}^+_\delta(\hat{\mathbf{p}},\hat{s})\hat{u}^+_\epsilon(\hat{\mathbf{j}}-\hat{\mathbf{p}},\hat{s})\hat{u}^+_\gamma(\hat{\mathbf{k}}-\hat{\mathbf{j}},\hat{s})\right\}\right]. \end{aligned} \tag{4.14}$$

Evaluating the CA of the first term on the RHS of equation (4.14) we see that it is zero because the CA will result in a expression similar to (4.12). The second term is linear in u^- as desired and we will come back to its evaluation. The last term will require a procedure similar to the one just done to obtain a term linear in u^-

$$\begin{aligned}
&\hat{\mathcal{P}}_{\mathcal{C}}\left\{\hat{u}_\delta^+(\hat{\mathbf{p}},\hat{s})\hat{u}_\epsilon^+(\hat{\mathbf{j}}-\hat{\mathbf{p}},\hat{s})\hat{u}_\gamma^+(\hat{\mathbf{k}}-\hat{\mathbf{j}},\hat{s})\right\} = \\
&= R_0\int_{-\infty}^{\hat{s}} d\hat{\tau}e^{-(\hat{\nu}_0(\hat{p})\hat{p}^2+\hat{\nu}_0(|\hat{\mathbf{j}}-\hat{\mathbf{p}}|)|\hat{\mathbf{j}}-\hat{\mathbf{p}}|^2+\hat{\nu}_0(|\hat{\mathbf{k}}-\hat{\mathbf{j}}|)|\hat{\mathbf{k}}-\hat{\mathbf{j}}|^2)(\hat{s}-\hat{\tau})}\times \\
&\times 3M^+_{\delta\rho\sigma}(\hat{\mathbf{p}})\int d^3q\left[\hat{\mathcal{P}}_{\mathcal{C}}\left\{\hat{u}_\rho^-(\hat{\mathbf{q}},\hat{\tau})\hat{u}_\sigma^-(\hat{\mathbf{p}}-\hat{\mathbf{q}},\hat{\tau})\hat{u}_\epsilon^+(\hat{\mathbf{j}}-\hat{\mathbf{p}},\hat{\tau})\hat{u}_\gamma^+(\hat{\mathbf{k}}-\hat{\mathbf{j}},\hat{\tau})\right\}\right. \\
&+2\hat{\mathcal{P}}_{\mathcal{C}}\left\{\hat{u}_\rho^-(\hat{\mathbf{q}},\hat{\tau})\hat{u}_\sigma^+(\hat{\mathbf{p}}-\hat{\mathbf{q}},\hat{\tau})\hat{u}_\epsilon^+(\hat{\mathbf{j}}-\hat{\mathbf{p}},\hat{\tau})\hat{u}_\gamma^+(\hat{\mathbf{k}}-\hat{\mathbf{j}},\hat{\tau})\right\} \\
&\left.+\hat{\mathcal{P}}_{\mathcal{C}}\left\{\hat{u}_\rho^+(\hat{\mathbf{q}},\hat{\tau})\hat{u}_\sigma^+(\hat{\mathbf{p}}-\hat{\mathbf{q}},\hat{\tau})\hat{u}_\epsilon^+(\hat{\mathbf{j}}-\hat{\mathbf{p}},\hat{\tau})\hat{u}_\gamma^+(\hat{\mathbf{k}}-\hat{\mathbf{j}},\hat{\tau})\right\}\right]. \qquad (4.15)
\end{aligned}$$

The first term on the RHS of this equation will be zero because when the CA is evaluated we find that

$$\begin{aligned}
&\hat{\mathcal{P}}_{\mathcal{C}}\left\{\hat{u}_\rho^-(\hat{\mathbf{q}},\hat{\tau})\hat{u}_\sigma^-(\hat{\mathbf{p}}-\hat{\mathbf{q}},\hat{\tau})\hat{u}_\epsilon^+(\hat{\mathbf{j}}-\hat{\mathbf{p}},\hat{\tau})\hat{u}_\gamma^+(\hat{\mathbf{k}}-\hat{\mathbf{j}},\hat{\tau})\right\} \\
&= \hat{u}_\rho^-(\hat{\mathbf{q}},\hat{\tau})\hat{u}_\sigma^-(\hat{\mathbf{p}}-\hat{\mathbf{q}},\hat{\tau})\hat{\mathcal{P}}_{\mathcal{C}}\left\{\hat{u}_\epsilon^+(\hat{\mathbf{j}}-\hat{\mathbf{p}},\hat{\tau})\hat{u}_\gamma^+(\hat{\mathbf{k}}-\hat{\mathbf{j}},\hat{\tau})\right\} \\
&= \hat{u}_\rho^-(\hat{\mathbf{q}},\hat{\tau})\hat{u}_\sigma^-(\hat{\mathbf{p}}-\hat{\mathbf{q}},\hat{\tau})\left\langle\hat{u}_\epsilon^+(\hat{\mathbf{j}}-\hat{\mathbf{p}},\hat{\tau})\hat{u}_\gamma^+(\hat{\mathbf{k}}-\hat{\mathbf{j}},\hat{\tau})\right\rangle \\
&= \hat{u}_\rho^-(\hat{\mathbf{q}},\hat{\tau})\hat{u}_\sigma^-(\hat{\mathbf{p}}-\hat{\mathbf{q}},\hat{\tau})Q(|\hat{\mathbf{k}}-\hat{\mathbf{j}}|)P_{\epsilon\gamma}(|\hat{\mathbf{k}}-\hat{\mathbf{j}}|)\delta(\hat{\mathbf{k}}-\hat{\mathbf{p}}), \qquad (4.16)
\end{aligned}$$

where we have used the property of isotropic tensors (1.20) to reduce the second order moment. Equation (4.16) implies that $\mathbf{k}=\mathbf{p}$; however, by our partitioning restrictions $\mathbf{k}$ lies in the low wavenumber domain whilst $\mathbf{p}$ is in the high wavenumber domain, and thus this term will give zero.

The second term in equation (4.15) is linear in u^- as desired. The last term can be subjected to a similar procedure above and will yield a term linear in u^- and a fifth order term in u^+ which can be expanded again. This procedure can be continued *ad infinitum*. To handle this we must look at the original low wavenumber NSE that we started with. If we substitute equation (4.15) into (4.14), and equation (4.14) into (4.11), and write the resultant equation in

shorthand notation we obtain

$$\left(\frac{\partial}{\partial\hat{t}}+\hat{\nu}_0(\hat{k})\hat{k}^2\right)u_k^- = R_0M_k^-u_j^-u_{k-j}^- + R_0^2M_k^-M_j^+\hat{\mathcal{P}}_{\mathcal{C}}\left\{u_p^-u_{j-p}^+u_{k-j}^+\right\} + R_0^3M_k^-M_j^+M_p^+\hat{\mathcal{P}}_{\mathcal{C}}\left\{u_q^-u_{p-q}^+u_{j-p}^+u_{k-j}^+\right\}+\mathcal{O}(R_0^4), \quad (4.17)$$

where as in Chapter 2, all convolutions, time integrals, Cartesian indices and decorations have been suppressed in the shorthand notation to facilitate ease of reading. One can now see why the NSE was initially written in dimensionless form. $|\hat{u}^+|$ is always ≤ 1 and equation (4.17) is effectively an expansion in the local Reynolds number R_0. Since R_0 is small in the UV area of the spectrum we are operating in, this allows us to truncate the above expansion at some order in R_0 with an associated small error term representing the rest of the terms. For the purposes of the calculations in this thesis, we will be truncating at second order in R_0. This being done and going back to the full notation, we are left with the equation

$$\left(\frac{\partial}{\partial\hat{t}}+\hat{\nu}_0(\hat{k})\hat{k}^2\right)\hat{u}_\alpha^-(\hat{\mathbf{k}},\hat{t}) = R_0M_{\alpha\beta\gamma}^-(\hat{\mathbf{k}})\int d^3\hat{j}\,\hat{u}_\beta^-(\hat{\mathbf{j}},\hat{t})\hat{u}_\gamma^-(\hat{\mathbf{k}}-\hat{\mathbf{j}},\hat{t}) + R_0^2\int d^3\hat{j}\int d^3\hat{p}\int_{-\infty}^{\hat{t}} d\hat{s}e^{-(\hat{\nu}_0(\hat{j})\hat{j}^2+\hat{\nu}_0(|\hat{\mathbf{k}}-\hat{\mathbf{j}}|)|\hat{\mathbf{k}}-\hat{\mathbf{j}}|^2)(\hat{t}-\hat{s})}\times \times 4M_{\alpha\beta\gamma}^-(\hat{\mathbf{k}})M_{\beta\delta\epsilon}^+(\hat{\mathbf{j}})\hat{\mathcal{P}}_{\mathcal{C}}\left\{\hat{u}_\delta^-(\hat{\mathbf{p}},\hat{s})\hat{u}_\epsilon^+(\hat{\mathbf{j}}-\hat{\mathbf{p}},\hat{s})\hat{u}_\gamma^+(\hat{\mathbf{k}}-\hat{\mathbf{j}},\hat{s})\right\}. \quad (4.18)$$

Evaluating the CA in equation (4.18) as done earlier in Chapter 4, we have

$$\begin{aligned}
&\hat{\mathcal{P}}_{\mathcal{C}}\left\{\hat{u}_\delta^-(\hat{\mathbf{p}},\hat{s})\hat{u}_\epsilon^+(\hat{\mathbf{j}}-\hat{\mathbf{p}},\hat{s})\hat{u}_\gamma^+(\hat{\mathbf{k}}-\hat{\mathbf{j}},\hat{s})\right\}\\
&=\hat{u}_\delta^-(\hat{\mathbf{p}},\hat{s})\hat{\mathcal{P}}_{\mathcal{C}}\left\{\hat{u}_\epsilon^+(\hat{\mathbf{j}}-\hat{\mathbf{p}},\hat{s})\hat{u}_\gamma^+(\hat{\mathbf{k}}-\hat{\mathbf{j}},\hat{s})\right\}\\
&=\hat{u}_\delta^-(\hat{\mathbf{p}},\hat{s})\left\langle\hat{u}_\epsilon^+(\hat{\mathbf{j}}-\hat{\mathbf{p}},\hat{s})\hat{u}_\gamma^+(\hat{\mathbf{k}}-\hat{\mathbf{j}},\hat{s})\right\rangle\\
&=\hat{u}_\delta^-(\hat{\mathbf{p}},\hat{s})\hat{Q}^+(|\hat{\mathbf{k}}-\hat{\mathbf{j}}|)P_{\epsilon\gamma}^+(\hat{\mathbf{k}}-\hat{\mathbf{j}})\delta(\hat{\mathbf{k}}-\hat{\mathbf{p}}),
\end{aligned} \quad (4.19)$$

where in the last line we have used the property of isotropic tensors (1.20), and the spectral density function Q is time independent because we are assuming stationarity. When equation (4.19) is substituted back into (4.18) and

the integration over $\mathbf{p}$ is evaluated, equation (4.18) becomes

$$\left(\frac{\partial}{\partial\hat{t}}+\hat{\nu}_0(\hat{k})\hat{k}^2\right)\hat{u}^-_\alpha(\hat{\mathbf{k}},\hat{t}) = R_0 M^-_{\alpha\beta\gamma}(\hat{\mathbf{k}})\int d^3\hat{j}\,\hat{u}^-_\beta(\hat{\mathbf{j}},\hat{t})\hat{u}^-_\gamma(\hat{\mathbf{k}}-\hat{\mathbf{j}},\hat{t}) + \\ +R_0^2\int d^3\hat{j}\int_{-\infty}^{\hat{t}} d\hat{s}e^{-(\hat{\nu}_0(\hat{j})\hat{j}^2+\hat{\nu}_0(|\hat{\mathbf{k}}-\hat{\mathbf{j}}|)|\hat{\mathbf{k}}-\hat{\mathbf{j}}|^2)(\hat{t}-\hat{s})}\times \\ \times 4M^-_{\alpha\beta\gamma}(\hat{\mathbf{k}})M^+_{\beta\delta\epsilon}(\hat{\mathbf{j}})P^+_{\epsilon\gamma}(\hat{\mathbf{k}}-\hat{\mathbf{j}})\hat{Q}^+(|\hat{\mathbf{k}}-\hat{\mathbf{j}}|)\hat{u}^-_\delta(\hat{\mathbf{k}},\hat{s}). \quad (4.20)$$

The isotropic tensors in the above equation may be handled by the relation

$$B_{\alpha\delta}(\mathbf{k}) = \frac{1}{d-1}\mathrm{Tr}(B_{\alpha\delta}(\mathbf{k}))P_{\alpha\delta}(\mathbf{k}), \quad (4.21)$$

where $B_{\alpha\delta}(\mathbf{k})$ is a second-rank isotropic tensor, d is the dimensionality of the system being studied (3 in our case) and $P_{\alpha\delta}(\mathbf{k})$ is given by (1.12). Relation (4.21) can be obtained from a property of matrices with rotational invariance [73, 62]. Using this relation we can write the pertinent terms in the integrand of equation (4.20) as

$$\begin{aligned} 4M^-_{\alpha\beta\gamma}(\hat{\mathbf{k}})M^+_{\beta\delta\epsilon}(\hat{\mathbf{j}})P^+_{\epsilon\gamma}(\hat{\mathbf{k}}-\hat{\mathbf{j}})\hat{u}^-_\delta(\hat{\mathbf{k}},\hat{s}) &= -L(\hat{\mathbf{k}},\hat{\mathbf{j}})P_{\alpha\delta}(\hat{\mathbf{k}})\hat{u}^-_\delta(\hat{\mathbf{k}},\hat{s}) \\ &= -L(\hat{\mathbf{k}},\hat{\mathbf{j}})\hat{u}^-_\alpha(\hat{\mathbf{k}},\hat{s}), \end{aligned} \quad (4.22)$$

where

$$\begin{aligned} L(\hat{\mathbf{k}},\hat{\mathbf{j}}) &= -2M^-_{\delta\beta\gamma}(\hat{\mathbf{k}})M^+_{\beta\delta\epsilon}(\hat{\mathbf{j}})P^+_{\epsilon\gamma}(\hat{\mathbf{k}}-\hat{\mathbf{j}}) \\ &= \frac{\left[\mu\left(\hat{k}^2+\hat{j}^2\right)-\hat{k}\hat{j}\left(1+2\mu^2\right)\right]\left(\mu^2-1\right)\hat{k}\hat{j}}{\hat{k}^2+\hat{j}^2-2\hat{k}\hat{j}\mu}, \end{aligned} \quad (4.23)$$

μ is the cosine of the angle between the $\mathbf{k}$ and $\mathbf{j}$ wave vectors, and the reason for the -ve signs is to do with this term being an increment to the viscosity which is on the LHS of the NSE. With this simplification, our equation (4.20) is now

$$\left(\frac{\partial}{\partial\hat{t}}+\hat{\nu}_0(\hat{k})\hat{k}^2\right)\hat{u}^-_\alpha(\hat{\mathbf{k}},\hat{t}) = R_0 M^-_{\alpha\beta\gamma}(\hat{\mathbf{k}})\int d^3\hat{j}\,\hat{u}^-_\beta(\hat{\mathbf{j}},\hat{t})\hat{u}^-_\gamma(\hat{\mathbf{k}}-\hat{\mathbf{j}},\hat{t}) \\ -\left\{R_0^2\int d^3\hat{j}\int_{-\infty}^{\hat{t}} d\hat{s}e^{-(\hat{\nu}_0(\hat{j})\hat{j}^2+\hat{\nu}_0(|\hat{\mathbf{k}}-\hat{\mathbf{j}}|)|\hat{\mathbf{k}}-\hat{\mathbf{j}}|^2)(\hat{t}-\hat{s})}\times \right. \\ \left. \times L(\hat{\mathbf{k}},\hat{\mathbf{j}})\hat{Q}^+(|\hat{\mathbf{k}}-\hat{\mathbf{j}}|)\hat{u}^-_\alpha(\hat{\mathbf{k}},\hat{s})\right\}. \quad (4.24)$$

This now only leaves the time integration to be evaluated.

4.2.1 Summing the time-history integral

The procedure for dealing with the time integral will essentially be the same as done in [64], prior to which a Markovian approximation used to be made (see [62, 5]). We write the time dependent terms in equation (4.24) as

$$\begin{aligned} R_0^2 \int_{-\infty}^{\hat{t}} d\hat{s} e^{-(\hat{\nu}_0(\hat{j})\hat{j}^2 + \hat{\nu}_0(|\hat{\mathbf{k}}-\hat{\mathbf{j}}|)|\hat{\mathbf{k}}-\hat{\mathbf{j}}|^2)(\hat{t}-\hat{s})} \hat{u}_\alpha^-(\hat{\mathbf{k}},\hat{s}) \\ = R_0^2 \int_0^\infty d\hat{\tau} e^{-\hat{\omega}_2(\hat{j},|\hat{\mathbf{k}}-\hat{\mathbf{j}}|)\hat{\tau}} \hat{u}_\alpha^-(\hat{\mathbf{k}},\hat{t}-\hat{\tau}) \;\; = \;\; I(\hat{\mathbf{k}},\hat{t}) \;, \end{aligned} \tag{4.25}$$

where we have made a change of variables $\hat{\tau} = \hat{t} - \hat{s}$ and $\hat{\omega}_2(\hat{j},|\hat{\mathbf{k}}-\hat{\mathbf{j}}|) = \hat{\nu}_0(\hat{j})\hat{j}^2 + \hat{\nu}_0(|\hat{\mathbf{k}}-\hat{\mathbf{j}}|)|\hat{\mathbf{k}}-\hat{\mathbf{j}}|^2$. Expanding $\hat{u}_\alpha^-(\hat{\mathbf{k}},\hat{t}-\hat{\tau})$ in a Taylor series around $\hat{\tau}=0$ we have

$$\hat{u}_\alpha^-(\hat{\mathbf{k}},\hat{t}-\hat{\tau}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} A_n(\hat{\mathbf{k}},\hat{t})\hat{\tau}^n, \tag{4.26}$$

where

$$A_n(\hat{\mathbf{k}},\hat{t}) = \frac{\partial^n}{\partial \hat{s}^n} \hat{u}_\alpha^-(\hat{\mathbf{k}},\hat{s})\bigg|_{\hat{s}=\hat{t}}. \tag{4.27}$$

Substituting equation (4.26) into (4.25) we get

$$I(\hat{\mathbf{k}},\hat{t}) = R_0^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} A_n(\hat{\mathbf{k}},\hat{t}) \int_0^\infty d\hat{\tau} e^{-\hat{\omega}_2(\hat{j},|\hat{\mathbf{k}}-\hat{\mathbf{j}}|)\hat{\tau}} \hat{\tau}^n. \tag{4.28}$$

Using the integral identity

$$\int d\tau e^{-G\tau}\tau^n = (-1)^n \frac{\partial^n}{\partial G^n} \int d\tau e^{-G\tau}, \tag{4.29}$$

we can write equation (4.28) as

$$\begin{aligned} I(\hat{\mathbf{k}},\hat{t}) &= R_0^2 \sum_{n=0}^{\infty} \frac{A_n(\hat{\mathbf{k}},\hat{t})}{n!} \frac{\partial^n}{\partial \hat{\omega}_2^n} \int_0^\infty d\hat{\tau} e^{-\hat{\omega}_2(\hat{j},|\hat{\mathbf{k}}-\hat{\mathbf{j}}|)\hat{\tau}} \\ &= R_0^2 \sum_{n=0}^{\infty} \frac{A_n(\hat{\mathbf{k}},\hat{t})}{n!} \frac{\partial^n}{\partial \hat{\omega}_2^n} \left[\hat{\omega}_2(\hat{j},|\hat{\mathbf{k}}-\hat{\mathbf{j}}|)^{-1}\right] \\ &= R_0^2 \sum_{n=0}^{\infty} (-1)^n A_n(\hat{\mathbf{k}},\hat{t}) \hat{\omega}_2(\hat{j},|\hat{\mathbf{k}}-\hat{\mathbf{j}}|)^{-(n+1)}, \end{aligned} \tag{4.30}$$

where in the last line we have used the identity

$$\frac{\partial^n}{\partial x^n}(x^{-1}) = (-1)^n n! x^{-(n+1)}. \tag{4.31}$$

To evaluate $A_n(\hat{\mathbf{k}}, \hat{t})$ in equation (4.30) we can use the NSE to see that

$$\frac{\partial}{\partial \hat{s}} \hat{u}_\alpha^-(\hat{\mathbf{k}}, \hat{s})\bigg|_{\hat{s}=\hat{t}} = -\hat{\nu}_0(\hat{k})\hat{k}^2 \hat{u}_\alpha^-(\hat{\mathbf{k}}, \hat{t}) + NLT + F, \tag{4.32}$$

where NLT is the non-linear term in the NSE and F is the forcing term. Differentiating again we see that

$$\begin{aligned}
\frac{\partial^2}{\partial \hat{s}^2} \hat{u}_\alpha^-(\hat{\mathbf{k}}, \hat{s})\bigg|_{\hat{s}=\hat{t}} &= -\hat{\nu}_0(\hat{k})\hat{k}^2 \frac{\partial}{\partial \hat{s}} \hat{u}_\alpha^-(\hat{\mathbf{k}}, \hat{s})\bigg|_{\hat{s}=\hat{t}} + \frac{\partial}{\partial \hat{s}} NLT\bigg|_{\hat{s}=\hat{t}} + \frac{\partial}{\partial \hat{s}} F\bigg|_{\hat{s}=\hat{t}} \\
&= (-1)^2 \left(\hat{\nu}_0(\hat{k})\hat{k}^2\right)^2 \hat{u}_\alpha^-(\hat{\mathbf{k}}, \hat{t}) + -\hat{\nu}_0(\hat{k})\hat{k}^2 F + \mathcal{O}(R_0) \\
&\approx (-1)^2 \left(\hat{\nu}_0(\hat{k})\hat{k}^2\right)^2 \hat{u}_\alpha^-(\hat{\mathbf{k}}, \hat{t}),
\end{aligned} \tag{4.33}$$

where in the second line we have substituted equation (4.32) for the velocity differentiation and evaluated the time differentiation of the force to be zero as the turbulence is stationary. Anything which involves the NLT we have ignored because when we substitute the time integral, that we are evaluating, back into the Reynolds term the NLT will just give us terms of order R_0^3 and we are truncating all our expansions at order R_0^2. In the final line we have ignored the effect of forcing as we are assuming the forcing to be confined to small wavenumbers such that the product $\hat{k}^2 F$ will be negligible. With these approximations we can write

$$\frac{\partial^n}{\partial \hat{s}^n} \hat{u}_\alpha^-(\hat{\mathbf{k}}, \hat{s})\bigg|_{\hat{s}=\hat{t}} \approx (-1)^n \left(\hat{\nu}_0(\hat{k})\hat{k}^2\right)^n \hat{u}_\alpha^-(\hat{\mathbf{k}}, \hat{t}), \tag{4.34}$$

and substituting this into equation (4.30) we get the result

$$\begin{aligned}
I(\hat{\mathbf{k}}, \hat{t}) &= R_0^2 \frac{\hat{u}_\alpha^-(\hat{\mathbf{k}}, \hat{t})}{\hat{\omega}_2(\hat{j}, |\hat{\mathbf{k}} - \hat{\mathbf{j}}|)} \sum_{n=0}^{\infty} \left[\frac{\hat{\nu}_0(\hat{k})\hat{k}^2}{\hat{\omega}_2(\hat{j}, |\hat{\mathbf{k}} - \hat{\mathbf{j}}|)}\right]^n \\
&= R_0^2 \frac{\hat{u}_\alpha^-(\hat{\mathbf{k}}, \hat{t})}{\hat{\nu}_0(\hat{j})\hat{j}^2 + \hat{\nu}_0(|\hat{\mathbf{k}} - \hat{\mathbf{j}}|)|\hat{\mathbf{k}} - \hat{\mathbf{j}}|^2} \times \\
&\quad \times \sum_{n=0}^{\infty} \left[\frac{\hat{\nu}_0(\hat{k})\hat{k}^2}{\hat{\nu}_0(\hat{j})\hat{j}^2 + \hat{\nu}_0(|\hat{\mathbf{k}} - \hat{\mathbf{j}}|)|\hat{\mathbf{k}} - \hat{\mathbf{j}}|^2}\right]^n.
\end{aligned} \tag{4.35}$$

If we note that

$$\frac{\hat{\nu}_0(\hat{k})\hat{k}^2}{\hat{\nu}_0(\hat{j})\hat{j}^2 + \hat{\nu}_0(|\hat{\mathbf{k}} - \hat{\mathbf{j}}|)|\hat{\mathbf{k}} - \hat{\mathbf{j}}|^2} < \frac{1}{2} \tag{4.36}$$

because $0 < \hat{k} < h$ and $h < j, |\mathbf{k} - \mathbf{j}| < 1$, and using the well-known geometric series sum

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \forall x < 1, \tag{4.37}$$

we can see that equation (4.35) becomes

$$\begin{aligned} I(\hat{\mathbf{k}}, \hat{t}) &= R_0^2 \frac{\hat{u}_\alpha^-(\hat{\mathbf{k}}, \hat{t})}{\hat{\nu}_0(\hat{j})\hat{j}^2 + \hat{\nu}_0(|\hat{\mathbf{k}} - \hat{\mathbf{j}}|)|\hat{\mathbf{k}} - \hat{\mathbf{j}}|^2} \times \\ &\quad \times \frac{1}{1 - \left[\hat{\nu}_0(\hat{k})\hat{k}^2 / \left(\hat{\nu}_0(\hat{j})\hat{j}^2 + \hat{\nu}_0(|\hat{\mathbf{k}} - \hat{\mathbf{j}}|)|\hat{\mathbf{k}} - \hat{\mathbf{j}}|^2\right)\right]} \\ &= R_0^2 \frac{\hat{u}_\alpha^-(\hat{\mathbf{k}}, \hat{t})}{\hat{\nu}_0(\hat{j})\hat{j}^2 + \hat{\nu}_0(|\hat{\mathbf{k}} - \hat{\mathbf{j}}|)|\hat{\mathbf{k}} - \hat{\mathbf{j}}|^2} \times \\ &\quad \times \frac{\hat{\nu}_0(\hat{j})\hat{j}^2 + \hat{\nu}_0(|\hat{\mathbf{k}} - \hat{\mathbf{j}}|)|\hat{\mathbf{k}} - \hat{\mathbf{j}}|^2}{\hat{\nu}_0(\hat{j})\hat{j}^2 + \hat{\nu}_0(|\hat{\mathbf{k}} - \hat{\mathbf{j}}|)|\hat{\mathbf{k}} - \hat{\mathbf{j}}|^2 - \hat{\nu}_0(\hat{k})\hat{k}^2} \\ &= R_0^2 \frac{\hat{u}_\alpha^-(\hat{\mathbf{k}}, \hat{t})}{\hat{\nu}_0(\hat{j})\hat{j}^2 + \hat{\nu}_0(|\hat{\mathbf{k}} - \hat{\mathbf{j}}|)|\hat{\mathbf{k}} - \hat{\mathbf{j}}|^2 - \hat{\nu}_0(\hat{k})\hat{k}^2}. \end{aligned} \tag{4.38}$$

Substituting this result for the time-integral back into our original equation (4.24) we get our final result

$$\begin{aligned} \left(\frac{\partial}{\partial \hat{t}} + \hat{\nu}_0(\hat{k})\hat{k}^2\right) \hat{u}_\alpha^-(\hat{\mathbf{k}}, \hat{t}) = R_0 M_{\alpha\beta\gamma}^-(\hat{\mathbf{k}}) \int d^3\hat{j}\, \hat{u}_\beta^-(\hat{\mathbf{j}}, \hat{t}) \hat{u}_\gamma^-(\hat{\mathbf{k}} - \hat{\mathbf{j}}, \hat{t}) - \\ -R_0^2 \int d^3\hat{j} \frac{L(\hat{\mathbf{k}}, \hat{\mathbf{j}})\hat{Q}^+(|\hat{\mathbf{k}} - \hat{\mathbf{j}}|)}{\hat{\nu}_0(\hat{j})\hat{j}^2 + \hat{\nu}_0(|\hat{\mathbf{k}} - \hat{\mathbf{j}}|)|\hat{\mathbf{k}} - \hat{\mathbf{j}}|^2 - \hat{\nu}_0(\hat{k})\hat{k}^2} \hat{u}_\alpha^-(\hat{\mathbf{k}}, \hat{t}). \end{aligned} \tag{4.39}$$

One should note that this effect of the time integral is different from the calculation presented by McComb and Johnston [64]. This is because of an error in the calculation of the latter due to not writing the viscosity as a function of wavenumber. McComb and Johnston had the last term on the RHS of equation (4.39) as

$$R_0^2 \int d^3\hat{j} \frac{L(\hat{\mathbf{k}}, \hat{\mathbf{j}})\hat{Q}^+(|\hat{\mathbf{k}} - \hat{\mathbf{j}}|)}{\hat{\nu}_0(\hat{j})\hat{j}^2 + \hat{\nu}_0(|\hat{\mathbf{k}} - \hat{\mathbf{j}}|)|\hat{\mathbf{k}} - \hat{\mathbf{j}}|^2} \left[\frac{(k^2/2) + j^2 - kj\mu}{j^2 - kj\mu}\right] \hat{u}_\alpha^-(\hat{\mathbf{k}}, \hat{t}), \tag{4.40}$$

whereas we have calculated the effect of the time-integral as putting another lifetime in the denominator of the viscosity increment, instead of the factor in square brackets.

4.3 Rescaling the equations

After the first shell removal, which is the first part of the RG algorithm, we obtained the coarse-grained NSE

$$\left(\frac{\partial}{\partial \hat{t}} + \hat{\nu}_1(\hat{k})\hat{k}^2\right) \hat{u}^-_\alpha(\hat{\mathbf{k}}, \hat{t}) = R_0 M^-_{\alpha\beta\gamma}(\hat{\mathbf{k}}) \int d^3\hat{j}\, \hat{u}^-_\beta(\hat{\mathbf{j}}, \hat{t}) \hat{u}^-_\gamma(\hat{\mathbf{k}} - \hat{\mathbf{j}}, \hat{t}), \tag{4.41}$$

where

$$\hat{\nu}_1(\hat{k}) = \hat{\nu}_0(\hat{k}) + \delta\hat{\nu}_0(\hat{k}), \tag{4.42}$$

is the recursion relation and

$$\delta\hat{\nu}_0(\hat{k}) = \frac{1}{\hat{k}^2} R_0^2 \int d^3\hat{j} \frac{L(\hat{\mathbf{k}}, \hat{\mathbf{j}})\hat{Q}^+(|\hat{\mathbf{k}} - \hat{\mathbf{j}}|)}{\hat{\nu}_0(\hat{j})\hat{j}^2 + \hat{\nu}_0(|\hat{\mathbf{k}} - \hat{\mathbf{j}}|)|\hat{\mathbf{k}} - \hat{\mathbf{j}}|^2 - \hat{\nu}_0(\hat{k})\hat{k}^2} \tag{4.43}$$

is the increment to the viscosity. Equation (4.41) is defined over the domain $0 \leq \hat{k} \leq h$. All that is left now is to implement the second part of the RG algorithm which is to rescale the equations so that they are defined over the old domain again. To rescale our system we need it to be defined over the domain $0 \leq \hat{k} \leq 1$. To avoid confusion we will introduce a notation shift when we rescale. Accordingly, we introduce the new scaled variables

$$\mathbf{k}' = \hat{\mathbf{k}}/h, \tag{4.44}$$

$$t' = \hat{t}/\tau(h), \tag{4.45}$$

$$\psi_\alpha(\mathbf{k}', t') = \hat{u}^-_\alpha(\hat{\mathbf{k}}, \hat{t})/V(h), \tag{4.46}$$

and substitute them into equation (4.41). After a bit of manipulation we get

$$\left(\frac{\partial}{\partial t'} + \nu_1'(k')k'^2\right) \psi_\alpha(\mathbf{k}', t') = R_1 M_{\alpha\beta\gamma}(\mathbf{k}') \int d^3 j\, \psi_\beta(\mathbf{j}', t')\psi_\gamma(\mathbf{k}' - \mathbf{j}', t'), \tag{4.47}$$

which is defined over the interval $0 \leq k \leq 1$ as desired, and where

$$\nu_1'(k') = \hat{\nu}_1(\hat{k})h^2\tau(h) = \hat{\nu}_1(hk')h^2\tau(h), \tag{4.48}$$

and

$$R_1 = R_0 h^4 \tau(h) V(h). \tag{4.49}$$

Note that with use of substitutions from (4.1), equation (4.44) can be written

$$\begin{aligned} \mathbf{k}' = \hat{\mathbf{k}}/h &= \mathbf{k}/k_0 h \\ &= \mathbf{k}/k_1, \end{aligned} \tag{4.50}$$

so indicating that scaling the dimensionless NSE by a factor of h is identical to having made the NSE dimensionless by scaling it with respect to k_1 instead of k_0. Similarly with the time scaling we get

$$\begin{aligned} t' = \hat{t}/\tau(h) &= t/\tau(k_0)\tau(h) \\ &= t/\tau(k_1), \end{aligned} \tag{4.51}$$

and with the velocity scaling we get

$$\begin{aligned} \psi_\alpha(\mathbf{k}', t') = \hat{u}_\alpha(\hat{\mathbf{k}}, \hat{t})/V(h) &= u_\alpha(\mathbf{k}, t)/V(k_0)V(h) \\ &= u_\alpha(\mathbf{k}, t)/V(k_1). \end{aligned} \tag{4.52}$$

Using (4.48) we must also scale equation (4.42)

$$\begin{aligned} h^2\tau(h)\hat{\nu}_1(\hat{k}) &= h^2\tau(h)\hat{\nu}_0(\hat{k}) + h^2\tau(h)\delta\hat{\nu}_0(\hat{k}) \\ \nu_1'(k') &= \nu_0'(k') + \delta\nu_0'(k'), \end{aligned} \tag{4.53}$$

which immediately leads onto the rescaling of equation (4.43)

$$\delta\nu_0'(k') = \tau^2(h)h^5R_0^2\frac{1}{k'^2}\int d^3j' \frac{L(\mathbf{k}', \mathbf{j}')\hat{Q}^+(|\hat{\mathbf{k}} - \hat{\mathbf{j}}|)}{\nu_0'(j')j'^2 + \nu_0'(|\mathbf{k}' - \mathbf{j}'|)|\mathbf{k}' - \mathbf{j}'|^2 - \nu_0'(k')k'^2}, \tag{4.54}$$

where we have used (4.48) to scale the viscosities in the denominator. To scale $\hat{Q}^+(|\hat{\mathbf{k}} - \hat{\mathbf{j}}|)$ appropriately, we need to look at the expression from which

it originates

$$\left\langle \hat{u}_\epsilon^+(\hat{\mathbf{k}}-\hat{\mathbf{j}},t)\hat{u}_\sigma^+(\hat{\mathbf{p}},t)\right\rangle = \hat{Q}(|\hat{\mathbf{k}}-\hat{\mathbf{j}}|)P_{\epsilon\sigma}(\hat{\mathbf{k}}-\hat{\mathbf{j}})\delta(\hat{\mathbf{k}}-\hat{\mathbf{j}}+\hat{\mathbf{p}}). \tag{4.55}$$

Since $\left\langle \hat{u}_\epsilon^+(\hat{\mathbf{k}}-\hat{\mathbf{j}},t)\hat{u}_\sigma^+(\hat{\mathbf{p}},t)\right\rangle$ scales like V^2, the delta function $\delta(\hat{\mathbf{k}}-\hat{\mathbf{j}}+\hat{\mathbf{p}})$ scales as the inverse of its argument i.e. k^{-3} and the projector $P_{\epsilon\sigma}(\hat{\mathbf{k}}-\hat{\mathbf{j}})$ is dimensionless, we have the scaling

$$Q'^+(|\mathbf{k}'-\mathbf{j}'|) = \frac{\hat{Q}^+(|\hat{\mathbf{k}}-\hat{\mathbf{j}}|)}{V^2(h)h^3}. \tag{4.56}$$

This being so, the rescaled viscosity increment becomes

$$\begin{aligned}\delta\nu_0'(k') &= V^2(h)\tau^2(h)h^8R_0^2\frac{1}{k'^2}\int d^3j' \frac{L(\mathbf{k}',\mathbf{j}')Q'^+(|\mathbf{k}'-\mathbf{j}'|)}{\nu_0'(j')j'^2+\nu_0'(|\mathbf{k}'-\mathbf{j}'|)|\mathbf{k}'-\mathbf{j}'|^2-\nu_0'(k')k'^2} \\ &= R_1^2\frac{1}{k'^2}\int d^3j' \frac{L(\mathbf{k}',\mathbf{j}')Q'^+(|\mathbf{k}'-\mathbf{j}'|)}{\nu_0'(j')j'^2+\nu_0'(|\mathbf{k}'-\mathbf{j}'|)|\mathbf{k}'-\mathbf{j}'|^2-\nu_0'(k')k'^2},\end{aligned} \tag{4.57}$$

where the last line follows from equation (4.49). This formally completes the rescaling procedure. However, we still need to get something which is actually calculable; in particular we need to model the form for Q'. To do this we first note that Q is related to the energy spectrum by

$$E(k) = 4\pi k^2 Q(k), \tag{4.58}$$

so that we may write

$$\begin{aligned}Q'^+(|\mathbf{k}'-\mathbf{j}'|) &= \frac{Q^+(|\mathbf{k}-\mathbf{j}|)}{V^2(k_1)k_1^3} \\ &= \frac{E^+(|\mathbf{k}-\mathbf{j}|)}{4\pi|\mathbf{k}-\mathbf{j}|^2V^2(k_1)k_1^3}.\end{aligned} \tag{4.59}$$

However, as was found before in the previous studies of McComb & Watt, the assumption of asymptotic freedom as developed in the previous chapter i.e. that we can approximate the CA over the subgrid shell as a full ensemble average, is a bit too strong an assumption. A better assumption is to approximate the CA in the sub-grid shell as being a Taylor series about the ensemble average at $k_0 \equiv k_{max}$ i.e. the wavenumber beyond which modes are asymptotically

free. Assuming a first order Taylor series expansion about k_0 we obtain

$$Q^+(|\mathbf{k}-\mathbf{j}|) \approx Q^+(|\mathbf{k}-\mathbf{j}|=k_0) + (|\mathbf{k}-\mathbf{j}|-k_0)\left.\frac{\partial Q^+(|\mathbf{k}-\mathbf{j}|)}{\partial|\mathbf{k}-\mathbf{j}|}\right|_{|\mathbf{k}-\mathbf{j}|=k_0}. \quad (4.60)$$

Using this approximation we can rewrite equation (4.59) to give

$$\begin{aligned} Q'^+(|\mathbf{k}'-\mathbf{j}'|) &= \frac{E^+(|\mathbf{k}-\mathbf{j}|=k_0)}{4\pi k_0^2 V^2(k_1)k_1^3} + \\ &\quad + \frac{(|\mathbf{k}-\mathbf{j}|-k_0)}{4\pi V^2(k_1)k_1^3}\,\frac{\partial}{\partial|\mathbf{k}-\mathbf{j}|}\left(\frac{E^+(|\mathbf{k}-\mathbf{j}|)}{|\mathbf{k}-\mathbf{j}|^2}\right)\Bigg|_{|\mathbf{k}-\mathbf{j}|=k_0} \end{aligned} \quad (4.61)$$

For this to be tractable we still need an expression for the energy spectrum. We obtain this by noting that in our RG calculation the fixed point that concerns us is the non-trivial Kolmogorov scaling that is obtained when the RG procedure reaches the scale-free inertial range. This being so, we make the approximation of assuming the Kolmogorov form for the energy spectrum in the subgrid shell. Although this would start off being a poor approximation, we hope that as the RG algorithm proceeds and we get closer to the inertial range fixed point, this approximation would become better. With this approximation, equation (4.61) now becomes

$$Q'^+(|\mathbf{k}'-\mathbf{j}'|) = \frac{\alpha\varepsilon^{2/3}}{4\pi V^2(k_1)k_1^3}\left[k_0^{-11/3} - (|\mathbf{k}-\mathbf{j}|-k_0)\,\frac{11}{3}k_0^{-14/3}\right], \quad (4.62)$$

which when made dimensionless in terms of the wavenumber $|\mathbf{k}'-\mathbf{j}'|$ by use of (4.50) we obtain

$$Q'^+(|\mathbf{k}'-\mathbf{j}'|) = \frac{\alpha\varepsilon^{2/3}k_1^{-20/3}}{4\pi V^2(k_1)}\left[h^{11/3} - \left(|\mathbf{k}'-\mathbf{j}'|-h^{-1}\right)\frac{11}{3}h^{14/3}\right]. \quad (4.63)$$

Substituting this into equation (4.57), and substituting for $R_1 = \tau(k_1)V(k_1)k_1^4$ we obtain

$$\delta\nu_0'(k') = \alpha\varepsilon^{2/3}k_1^{4/3}\tau^2(k_1)\frac{1}{4\pi k'^2}\int d^3j'\,\frac{L(\mathbf{k}',\mathbf{j}')\left[h^{11/3} - (|\mathbf{k}'-\mathbf{j}'|-h^{-1})\,\frac{11}{3}h^{14/3}\right]}{\nu_0'(j')j'^2 + \nu_0'(|\mathbf{k}'-\mathbf{j}'|)|\mathbf{k}'-\mathbf{j}'|^2 - \nu_0'(k')k'^2}. \quad (4.64)$$

For both sides of equation (4.64) to be consistent i.e. both in terms of dimensionless variables, we require that

$$\tau(k_1) = \alpha^{-1/2}\varepsilon^{-1/3}k_1^{-2/3}, \quad (4.65)$$

which fixes the form of the time scaling parameter τ. The dimensionless constant of α is included as a self-consistent measure of having the viscosity increment scale in the same way as the viscosity; as it should do (see [62]). Another reason for eliminating α in equation (4.64) is that we do not want to assume a value of α everytime we calculate the viscosity increment in our RG calculation.

We can also now see more clearly the way that the viscosity scales by substituting (4.65) in equation (4.6)

$$\hat{\nu}_0(\hat{k}) = \alpha^{-1/2}\varepsilon^{-1/3}k_0^{4/3}\nu_0(k), \tag{4.66}$$

such that if we wanted to scale the viscosity on the scale k_1 instead, we would have the scaling relation

$$\nu_0'(k') = \alpha^{-1/2}\varepsilon^{-1/3}k_1^{4/3}\nu_0(k). \tag{4.67}$$

Furthermore, we can relate $\nu_0'(k')$ to $\hat{\nu}_0(\hat{k})$ by using the fact that $k_1 = hk_0$ in (4.67)

$$\begin{aligned}
\nu_0'(k') &= \alpha^{-1/2}\varepsilon^{-1/3}k_1^{4/3}\nu_0(k) \\
&= \alpha^{-1/2}\varepsilon^{-1/3}h^{4/3}k_0^{4/3}\nu_0(k) \\
&= h^{4/3}\hat{\nu}_0(\hat{k}) \\
&= h^{4/3}\hat{\nu}_0(hk'),
\end{aligned} \tag{4.68}$$

where in the last line we have used the relation $\hat{k} = hk'$ which is a corollary of $k_1 = hk_0$, and where $\hat{\nu}_0(hk')$ is defined over a domain which is a factor of h less than the domain over which $\nu_0'(k')$ is defined. Thus relation (4.68) provides us with a scaling transformation to take a viscosity defined on a domain R, say, to a domain R' which is bigger by a factor of $1/h$.

4.4 The RG equations

If we analyse equation (4.53) we find that the LHS viscosity is defined on $0 < k' < 1$, whereas the viscosities on the RHS are defined on the domain $0 < k' < h^{-1}$. For the purposes of the RG recursive procedure we would like to

have all the viscosities defined over the same domain, which we prefer to be $0 < k' < 1$. Since (4.68) shows a way of formally reducing the domain of the viscosities whilst still retaining the scaling upon k_1 (denoted by the primes), it is a straight forward procedure to change the domain of the viscosities on the RHS of equation (4.53) to get

$$\nu_1'(k') = h^{4/3}\hat{\nu}_0(hk') + h^{-4/3}\delta\hat{\nu}_0(k'), \tag{4.69}$$

where the factor $h^{-4/3}$ comes from the viscosities in the denominator of $\delta\nu_0'(k')$. Now that we have all the viscosities, on both RHS and LHS, defined over the same domains $0 < k' < 1$, we need to do some relabeling. Remember that the primes and hats were only labels to distinguish domain sizes where the viscosities live. Since all the viscosities now live on the same domain, they should all have the same label. Thus we will make all the viscosity labeling uniform and label all by a prime; this being equivalent to $\hat{} , \ ' \rightarrow \ '$. Making this change of labeling and generalising equation (4.69) to an iteration from a viscosity indexed by n to one indexed by $n+1$ we get

$$\nu_{n+1}'(k') = h^{4/3}\nu_n'(hk') + h^{-4/3}\delta\nu_n'(k'), \tag{4.70}$$

where $\delta\nu_n'(k')$ is now given by

$$\delta\nu_n'(k') = \frac{1}{4\pi k'^2}\int d^3j' \frac{L(\mathbf{k}',\mathbf{j}')\left[h^{11/3} - (|\mathbf{k}'-\mathbf{j}'| - h^{-1})\,\frac{11}{3}h^{14/3}\right]}{\nu_n'(hj')j'^2 + \nu_n'(h\,|\mathbf{k}'-\mathbf{j}'|)\,|\mathbf{k}'-\mathbf{j}'|^2 - \nu_n'(hk')k'^2}. \tag{4.71}$$

Equations (4.70) and (4.71) are the RG recursion equations which we use to iterate the RG mode elimination procedure until we obtain a fixed point in the effective viscosity. At the fixed point we obtain the *scale-invariant* renormalized viscosity, ν'^*, which occurs when

$$\nu_n' = \nu_{n+1}' \equiv \nu'^*. \tag{4.72}$$

4.5 Results

Before we begin to show some of the results of the RG procedure we must say something about the local Reynolds number as the RG algorithm proceeds. A major premise of the RG calculation above is that the local Reynolds number

is small so that one can use it to truncate the expansions in the calculation. We can show that it stays small as the RG iteration proceeds by looking at its upper limit. Recall that for an arbitrary wavenumber κ we have the local Reynolds number

$$R(\kappa) = \tau(\kappa)V(\kappa)\kappa^4. \tag{4.73}$$

To obtain an expression for $V(\kappa)$ we look back at its definition in equation (4.4) and reduce the second order moment to the spectral density function using the property of isotropic tensors

$$\begin{aligned} V^2(\kappa) &= \frac{1}{\kappa^3}\int d^3j \, \langle u_\alpha(\boldsymbol{\kappa},t)u_\alpha(\mathbf{j},t)\rangle \\ &= \frac{1}{\kappa^3}\int d^3j Q(\kappa)P_{\alpha\alpha}(\boldsymbol{\kappa})\delta(\mathbf{k}-\mathbf{j}) \\ &= \frac{2Q(\kappa)}{\kappa^3}, \end{aligned} \tag{4.74}$$

where we have used $P_{\alpha\alpha}(\boldsymbol{\kappa}) = 2$. We can now use (4.58) to put this in the form of the energy spectrum

$$V^2(\kappa) = \frac{2E(\kappa)}{4\pi\kappa^5}. \tag{4.75}$$

Substituting this into (4.73) along with the form for $\tau(\kappa)$ given by (4.65) we get

$$R(\kappa) = \alpha^{-1/2}\varepsilon^{-1/3}\kappa^{5/6}\sqrt{\frac{E(\kappa)}{2\pi}}. \tag{4.76}$$

At the RG fixed point we expect to have reached the inertial range with the Kolmogorov form for the energy spectrum. We can regard this as an upper limit for the energy spectrum that will be reached in our calculation as the spectrum will definitely be less than this in the higher wavenumbers which we will be passing through the course of the RG procedure. Accordingly, we can use the Kolmogorov spectrum to provide us with an upper limit on what value the local Reynolds number will take. Substituting the Kolmogorov spectrum in (4.76) we get the upper limit

$$R(\kappa) \leq \sqrt{\frac{1}{2\pi}}, \tag{4.77}$$

which shows that the local Reynolds number does indeed stay small i.e. less than one, throughout the RG iteration.

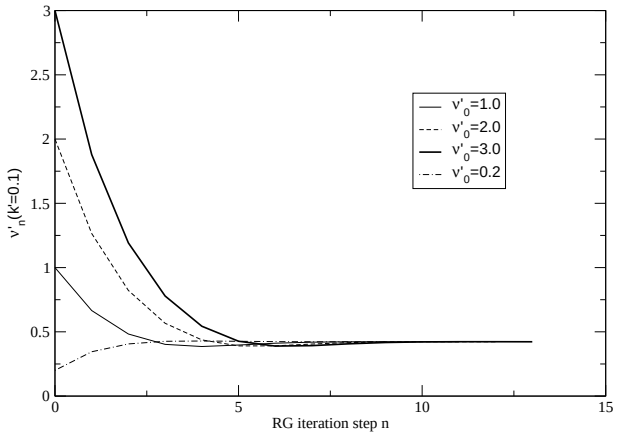

Figure 4.1: RG iteration map showing the convergence of the renormalized viscosity ν' at a wavenumber $k' = 0.1$ for several initial viscosities. A shell width of $\eta = 0.3$ has been used for the above case.

4.5.1 Renormalized fixed point viscosity

When equations (4.70) and (4.71) are iterated we find that the calculation converges upon a fixed point in the renormalized viscosity. The evolution of the renormalized viscosity and convergence upon a fixed point can be seen in the RG map of fig. 4.1. The calculation converges to the same fixed point for several values of the initial scaled viscosity ν'_0, thus illustrating the universal nature of this value. This area of universality shows that the fixed point is independent of the details of the system i.e. initial and boundary conditions, and only depends upon the behaviour of the inertial range where scale is not important. Thus the behaviour of the fixed point quantities are valid for all systems independent of the details of the system, as long as the Reynolds number is high enough for an inertial range to exist.

Fig. 4.2 shows the fixed point renormalized viscosity as a function of the scaled wavenumber for various values of the shell width η. Note that for large values of η the fixed point renormalized viscosity has negligible dependence upon wavenumber. One should also note that in the limit $k' \to 0$ the fixed point renormalized viscosity tends to a constant value. Thus the apparent dangerously divergent looking k'^2 in the denominator of equation (4.71) for the incre-

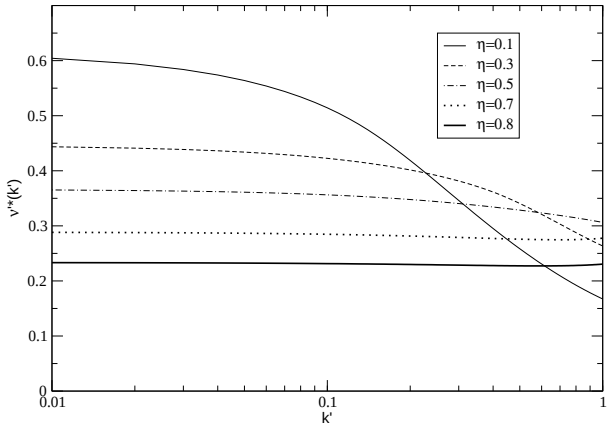

Figure 4.2: Wavenumber dependent fixed point renormalized viscosities, $\nu'^*(k')$, for various coarse-graining shell widths, η.

ment to the renormalized viscosity can be shown to be a singularity which is integrable (see Storkey [74]). The behaviour of the renormalized viscosity in both of these cases can be explained by the notion of *scale separation* which is a recurring theme in the first part of this thesis. If we believe the notion of localness in wavenumber/scale of the energy transfer process put forward in the Kolmogorov hypotheses, we will expect that scales which are sufficiently separate will be independent of the way we account for them. So to explain further, we can see from fig. 4.2 that the asymptotic nature of the renormalized viscosity to tend to a constant value for small wavenumbers can be attributed to the fact that the small wavenumbers which are far from the subgrid scales, cannot see the phase details of the subgrid scales; they see only the 'bigger' picture of energy dissipation and transfer; so they see all the sub-grid scales behaving essentially in the same way. This behaviour should persist irrespective of whether we have an exact deterministic picture of the subgrid scales or just a mean-field averaged one like we have here. On the other hand, the wavenumbers which are near the cut-off of the resolved scales are 'local' in the neighbourhood of the subgrid-scales and so are more attuned with the subgrid phase. If, however, we increase the shell width η, then we have a scenario where most of the subgrid phase is still 'screened' from the cut-off wavenumbers in the resolved band and only the wavenumbers close to the cut-off in the

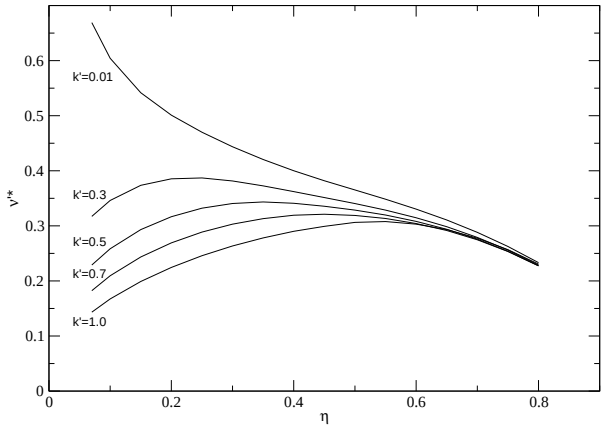

Figure 4.3: Graph showing the fixed point renormalized viscosity values ν'^* Vs. the shell width η, for various values of the scaled wavenumber k'.

subgrid shell will have their phase details seen. However, if we coarse-grain our subgrid shell based on the details of the shell near the asymptotically free end, and we have a large enough shell width, then the wavenumbers near the cut-off in the resolved band will still be ignorant of the phase details of the whole of the subgrid shell; thus accounting for our result of the renormalized viscosity being wavenumber independent for large η. If we had also coarse-grained our subgrid shell based on the details near the cut-off, we should find that this wavenumber independence of the renormalized viscosity would disappear. The importance of scale separation and the ideas and discussion presented above are manifested most in fig. 4.3 where we can quite clearly see that for small η we have a wide scale separation in the fixed point renormalized viscosity, whilst at large η the scales all have similar fixed point renormalized viscosity values.

Fig. 4.4 shows the evolution and wavenumber behaviour of the actual eddy viscosity though the RG iteration. This is what would then be used in a large eddy simulation (LES) as a subgrid model along with its associate cut-off wavenumber. The eddy viscosity has been made dimensionless on the scale k_0 for convenience. The value of the eddy viscosity increases for each coarse-graining of the RG procedure, as it should do, to account for the energy dissipation of the missing subgrid scales. The coarse-graining also explains why the eddy

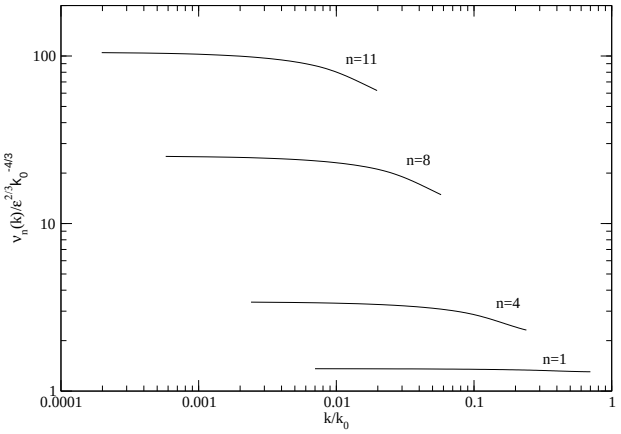

Figure 4.4: Evolution of the effective viscosity made dimensionless on k_0, for each RG iteration n. For the above case, a shell width of $\eta = 0.3$ has been used which reached the fixed point renormalized viscosity fixed point at $n = 11$.

viscosity is defined over a more restricted support at each stage of the RG iteration.

4.5.2 Evolution of k_d

Fig. 4.5 shows the evolution of the renormalized Kolmogorov dissipation wavenumber, $k_d^{(n)}$, through the RG iteration. This is obtained from the renormalized viscosity by modifying the usual definition of k_d to account for wavenumber dependent viscosities.

$$k_d^{(n)} = \left(\frac{\varepsilon}{\langle \nu_n'(k') \rangle_{k'}} \right)^{1/4}, \tag{4.78}$$

where $\langle\ \rangle_{k'}$ denotes an average over k' and in fig. 4.5 we have scaled $k_d^{(n)}$ on $k_d^{(0)}$ for the computational convenience of not having to input a value for ε. Fig. 4.6 shows the evolution of the actual Kolmogorov dissipation wavenumber based upon a similar relation to (4.78) but using the values for the actual eddy viscosities as shown in fig. 4.4. As expected the actual Kolmogorov dissipation wavenumber decreases as more and more scales are removed.

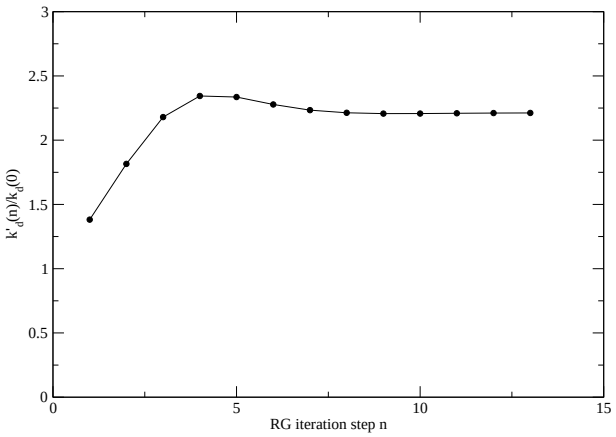

Figure 4.5: Evolution of the renormalized Kolmogorov dissipation wavenumber (scaled on $k_d(n=0)$) for $\eta = 0.3$, constructed using the renormalized viscosity at each stage of the RG iteration n.

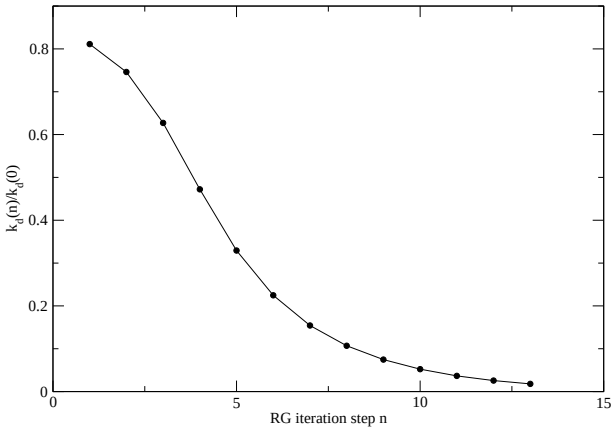

Figure 4.6: Evolution of the Kolmogorov dissipation wavenumber (scaled on $k_d(n=0)$) for $\eta = 0.3$ constructed using the effective viscosity made dimensionless on k_0 (see fig. 4.4).

4.5.3 Prediction of the Kolmogorov constant

If we look at the unscaled version of the dimensionless NSE (equation (4.47)), post RG, we will be presented with a relation such as the following

$$\left(\frac{\partial}{\partial t}+\nu^*(k)k^2\right)u_\alpha^-(\mathbf{k},t)=M_{\alpha\beta\gamma}^-(\mathbf{k})\int d^3j u_\beta^-(\mathbf{j},t)u_\gamma^-(\mathbf{k}-\mathbf{j},t)+f_\alpha^-(\mathbf{k},t), \quad (4.79)$$

where we have again included the forcing. This equation represents the resolved modes NSE where the average effect of the subgrid modes has enhanced the viscosity to make an effective viscosity, and where the highest wavenumber is in the top of the universal inertial range. We can construct an energy balance equation from (4.79) in a similar way to that done in Chapter 1. This gives an identical expression to (1.30) but defined over the resolved scales

$$2\nu^*(k)k^2E^-(k)=T^-(k)+W^-(k), \quad (4.80)$$

where the time derivative vanishes because we are assuming stationarity. When integrated over the domain $0\leq k\leq k_c$, where k_c is the cut-off wavenumber at the top of the domain of the resolved scales, the nonlinear term $T^-(k)$ vanishes (see Appendix B) and we get

$$\int_0^{k_c}dk2\nu^*(k)k^2E^-(k)=\int_0^{k_c}dkW^-(k)=\varepsilon, \quad (4.81)$$

where the last equality comes from the fact that we are assuming that the forcing is confined to the small wavenumbers such that

$$\int_0^{k_c}dkW^-(k)=\int_0^{k_0}dkW(k)=\varepsilon. \quad (4.82)$$

The next chapter will deal in more detail with the kinetics of turbulent transport and on relations such as these. Going back to equation (4.81) we can isolate the expression

$$\int_0^{k_c}dk2\nu^*(k)k^2E^-(k)=\varepsilon, \quad (4.83)$$

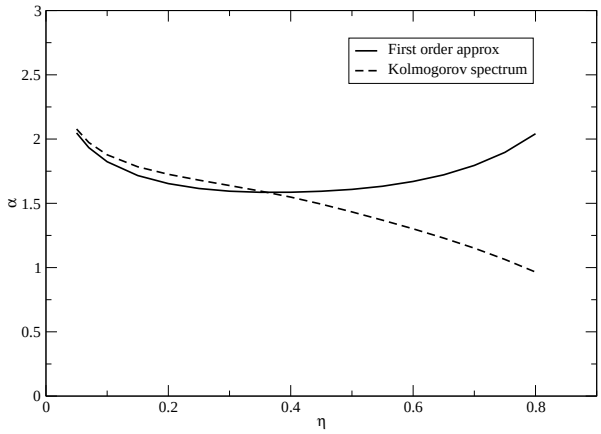

Figure 4.7: The Kolmogorov constant α Vs. the coarse-graining shell width parameter η. We have shown the results for assuming the Kolmogorov spectrum in the whole sub-grid shell as well as the results for taking the shell to be represented by a first order Taylor series approximation of the Kolmogorov spectrum around the high wavenumber end of the subgrid shell.

which when scaled on the cut-off wavenumber, k_c, and assuming the Kolmogorov form of the energy spectrum gives

$$1 = 2\alpha^{3/2} \int_0^1 dk' \nu'^*(k') k'^{1/3}. \tag{4.84}$$

We can assume the Kolmogorov form for the energy spectrum (valid at infinite Reynolds number) because our renormalized eddy viscosity is universal i.e. it applies to all Reynolds numbers high enough such that an inertial range exists. Finally, we can rearrange equation (4.84) to give us an expression for calculating the Kolmogorov constant from our renormalized viscosity

$$\alpha = \left\{ 2 \int_0^1 dk' \nu'^*(k') k'^{1/3} \right\}^{-2/3}. \tag{4.85}$$

Fig. 4.7 shows the results for the prediction of the Kolmogorov constant for various values of the shell width parameter η. We will be primarily concentrating on the results for the case of the first order Taylor series approximation of the Kolmogorov spectrum around the high wavenumber end of the subgrid shell

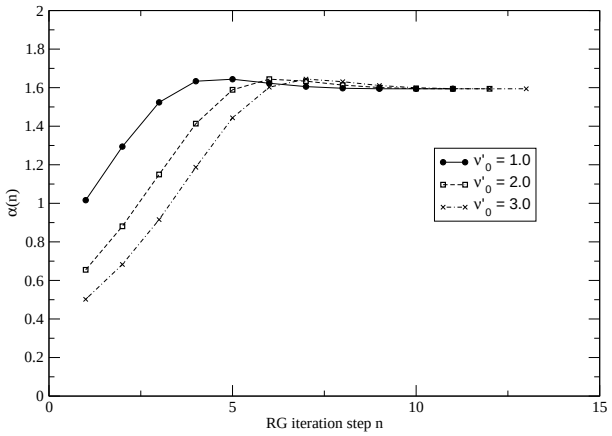

Figure 4.8: RG map showing the evolution and convergence of the Kolmogorov constant α through the RG iteration, for several values of the initial scaled viscosity ν_0'.

in equation (4.64) for the increment to the viscosity. The case of assuming the Kolmogorov spectrum throughout the shell is also shown for comparison. The first thing to note is that, remembering the RG invariance to a particular scale mentioned earlier in the introduction, we are looking for an area where the results are independent of the shell width η. We can see that the first order Taylor approximation shows an approximate plateau $0.2 \lesssim \eta \lesssim 0.5$ where we have α approximately independent from η. In this area α takes the value 1.62 ± 0.05 in very good agreement with current experiment [75] and computational [76] surveys. The tendency of α to increase in the limits of η going to 0 and 1 indicates the breakdown of the theory due to the increasing invalidity of our approximations. In the limit of $\eta \to 0$ where the subgrid shell is shrunk to zero, we need a more deterministic approximation for the subgrid modes, whilst in the limit $\eta \to 1$ our first order Taylor approximation will clearly not be representative over the whole subgrid shell.

Finally fig. 4.8 shows the RG map of the evolution of the Kolmogorov constant which, being based on the renormalized viscosity through equation (4.85), again (unsurprisingly) shows the universal nature of the fixed point. It is important to remember that it is the viscosity which is being renormalized and which reaches the fixed point, and not the wavenumber. The other renormal-

ized quantities such as the Kolmogorov constant α, the Kolmogorov dissipation wavenumber k_d, the dissipation parameter ε etc. are all obtained from the renormalized viscosity.

4.6 Comments

The RG procedure of McComb *et. al.* has been used for actual large eddy simulations (see Hunter [72], Johnston [77], and Verma & Kumar [78]) with results that are as good as any others in the field. The calculation has also been developed and extended to the study of magnetohydrodynamics [79].

A major criticism of the RG method of McComb *et. al.* is the neglect of the effects of cubic non-linearities which arise when including the effects of the cross-term in the low wavenumber NSE (4.9) as well as the Reynolds term. Recall that the Reynolds term was expanded to second order in the local Reynolds number R_0, whereas the cross term conditional average was just evaluated at first order. This was justified by saying that the cross-term was already in form linear in the velocity u^-. Possibly linked to this inconsistency is the lack of an upturn or 'cusp' like behaviour of the effective viscosity in the limit $k \to k_c$ which is seen clearly in numerical results [65, 80]. We hope to address these and other questions in the next chapter where we find that neglecting the cross-term has a significant effect on dissipation.

Chapter 5

Modeling energy dissipation in coarse-grained systems

In Chapter 2 we set down the criteria that any RG method should fulfil in order to be considered for being used in an LES model. Recall that although retaining all phase details between the subgrid and resolved velocity modes is in general a formidable task, we said that at least one should require that all the kinetics are preserved. This primarily concerned the conserving of total energy in the system and the rate of energy dissipation. In this chapter we will begin by very briefly reviewing the main results of turbulent energy transfer, and as before we shall be mainly restricting ourselves to the case of statistically stationary turbulence. We will then move onto discussing the kinetics of partitioned flows, as in the case of LES, RG and any other coarse-grained models. This will be focusing on the separate and composite behaviours of the 'cross' and 'Reynolds' interaction terms as defined in previous chapters, and will involve the reviewing of several numerical studies and presenting new analytic results. From this we will move onto discussing the implications that these results have for RG, and in particular the RG method presented in the previous chapter which does not include the effects of the cross term. A simple heuristic model for including these effects will then be proposed which is based on expanding the cross term to second order in local Reynolds number. The results of this study, although possessing some pathological problems, agree qualitatively with those from numerical simulations. Note that most of the conventions and many of the necessary proofs in this chapter can be found in Appendix B.

5.1 Summary of turbulent energy transfer

We begin as always with the incompressible NSE as given by equation (1.10). However, instead of the normal symmetric $M_{\alpha\beta\gamma}(\mathbf{k})$ operator (given by equation (1.11)), we will now choose to work with a non-symmetric (with respect to β and γ indices) operator $N_{\alpha\beta\gamma}(\mathbf{k})$ given by

$$N_{\alpha\beta\gamma}(\mathbf{k}) := -ik_\gamma P_{\alpha\beta}(\mathbf{k}) \ , \tag{5.1}$$

where $P_{\alpha\beta}(\mathbf{k})$ is given by equation (1.12). The equivalence of the NSE with M's instead of N's can be seen if we write out the NSE and expand the M operator

$$\begin{aligned}\left[\frac{\partial}{\partial t}+\nu k^2\right]u_\alpha(\mathbf{k},t) &= -i\frac{1}{2}k_\beta P_{\alpha\gamma}(\mathbf{k})\int d^3j u_\beta(\mathbf{j},t)u_\gamma(\mathbf{k}-\mathbf{j},t) \\ &\quad -i\frac{1}{2}k_\gamma P_{\alpha\beta}(\mathbf{k})\int d^3j u_\beta(\mathbf{j},t)u_\gamma(\mathbf{k}-\mathbf{j},t)+f_\alpha(\mathbf{k},t) \ , \end{aligned} \tag{5.2}$$

and focusing on the first term on the RHS, we rename dummy indices $\beta \leftrightarrow \gamma$ and interchange the dummy variables inside the convolution (see Appendix B) to get

$$\begin{aligned}\left[\frac{\partial}{\partial t}+\nu k^2\right]u_\alpha(\mathbf{k},t) &= -ik_\gamma P_{\alpha\beta}(\mathbf{k})\int d^3j u_\beta(\mathbf{j},t)u_\gamma(\mathbf{k}-\mathbf{j},t)+f_\alpha(\mathbf{k},t) \\ &= N_{\alpha\beta\gamma}(\mathbf{k})\int d^3j u_\beta(\mathbf{j},t)u_\gamma(\mathbf{k}-\mathbf{j},t)+f_\alpha(\mathbf{k},t) \ . \end{aligned} \tag{5.3}$$

This is the form of the NSE we will be using throughout this chapter. It is important to note that unlike the $M_{\alpha\beta\gamma}(\mathbf{k})$ form, the $N_{\alpha\beta\gamma}(\mathbf{k})$ form is not symmetric under the renaming of the β and γ indices. The reason for this change of the NSE will become clear later when illustrating the different effects from each of the cross terms on the transfer of energy across an arbitrary partition in k-space. From the definition of $M_{\alpha\beta\gamma}(\mathbf{k})$ in (1.11) and the definition of $N_{\alpha\beta\gamma}(\mathbf{k})$ in (5.1) it can easily be seen that

$$M_{\alpha\beta\gamma}(\mathbf{k}) = \frac{1}{2}\left[N_{\alpha\beta\gamma}(\mathbf{k})+N_{\alpha\gamma\beta}(\mathbf{k})\right] \ . \tag{5.4}$$

From equation (5.3) we can construct the spectral energy equation in a similar

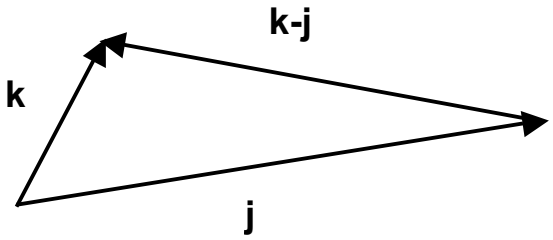

Figure 5.1: The triangle condition needed to satisfy homogeneity in velocity mode triad interactions.

way to equations (1.29)-(1.31)

$$\left[\frac{\partial}{\partial t} + 2\nu k^2\right] E(k,t) = T(k,t) + W(k,t) \ , \tag{5.5}$$

where $W(k,t)$ is the rate at which energy per wavenumber k, is being injected into the system, and where the non-linear transfer term $T(k,t)$ is now defined with the N operator

$$\begin{aligned} T(k,t) \ := \ 2\pi k^2 N_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j \, \{ & \langle u_\alpha(-\mathbf{k},t)\, u_\beta(\mathbf{j},t) u_\gamma(\mathbf{k}-\mathbf{j},t)\rangle \\ & - \langle u_\alpha(\mathbf{k},t)\, u_\beta(-\mathbf{j},t) u_\gamma(-\mathbf{k}+\mathbf{j},t)\rangle \} \ . \end{aligned} \tag{5.6}$$

The third order moments in definition (5.6), are known as triad interactions; so-called due to the occurrence of an interaction between velocity modes of three wavevectors. Homogeneity (see McComb [5] and Leslie [81]) implies that these three wavevectors should sum to zero; known as the *triangle condition*. If we write the first triad interaction term in definition (5.6) in the suggestive form

$$2\pi k^2 N_{\alpha\beta\gamma}(\mathbf{k}) \left\langle u_\alpha(-\mathbf{k},t) \int d^3j u_\beta(\mathbf{j},t) u_\gamma(\mathbf{k}-\mathbf{j},t) \right\rangle \tag{5.7}$$

then we will find it useful to be able to talk about the triad interaction as being the *average* (represented by angle brackets) of all two-velocity mode interactions (represented by the convolution of uu) transferring energy to or from a velocity mode corresponding to a particular wavevector $\mathbf{k}$ via the interaction *vertex* N.

5.1.1 Energy conservation

When integrated over all k-space, equation (5.5) becomes

$$\frac{\partial}{\partial t}E(t) = \varepsilon_w(t) - \varepsilon_d(t) \ , \tag{5.8}$$

where both $\varepsilon_w(t)$ and $\varepsilon_d(t)$ are defined as in equations (1.36) and (1.37), and $E(t) = \int dk E(k,t)$. The derivation of equation (5.8) relies on

$$\int_0^\infty dk T(k,t) = 0 \ , \tag{5.9}$$

which if we look at the explicit form of $T(k,t)$ from equation (5.6), and the details of the proof in Appendix B, we see that this essentially required that the integration limits of the two variables k and j be the same. From the result in equation (5.9), we can immediately see that the following result holds

$$\int_0^{k'} dk T(k,t) = -\int_{k'}^\infty dk T(k,t) \ , \tag{5.10}$$

for any arbitrary wavenumber k'.

5.2 Partitioned spectral energy equations

Most of the previous section was just a recap of section 1.4 in Chapter 1. However, it is essentially the description that we will now be taking over into partitioned systems, and the tools used to get the above forms will be the only ones necessary to get the following forms.

We start with partitioning the NSE parameterised by a wavenumber k_c, into

$$\begin{aligned}\left[\frac{\partial}{\partial t} + \nu_0 k^2\right] u_\alpha^-(\mathbf{k},t) \ &= \ N^-_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j \left\{ u_\beta^-(\mathbf{j},t) u_\gamma^-(\mathbf{k}-\mathbf{j},t) \right. \\ &\quad + u_\beta^-(\mathbf{j},t) u_\gamma^+(\mathbf{k}-\mathbf{j},t) + u_\beta^+(\mathbf{j},t) u_\gamma^-(\mathbf{k}-\mathbf{j},t) \\ &\quad \left. + u_\beta^+(\mathbf{j},t) u_\gamma^+(\mathbf{k}-\mathbf{j},t) \right\} \ , \end{aligned} \tag{5.11}$$

for low-k, $k \leq k_c$, modes and

$$\left[\frac{\partial}{\partial t} + \nu_0 k^2\right] u_\alpha^+(\mathbf{k},t) = N^+_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j \left\{ u_\beta^-(\mathbf{j},t) u_\gamma^-(\mathbf{k}-\mathbf{j},t) + u_\beta^-(\mathbf{j},t) u_\gamma^+(\mathbf{k}-\mathbf{j},t) + u_\beta^+(\mathbf{j},t) u_\gamma^-(\mathbf{k}-\mathbf{j},t) + u_\beta^+(\mathbf{j},t) u_\gamma^+(\mathbf{k}-\mathbf{j},t) \right\} , \quad (5.12)$$

for the high-k, $k \geq k_c$ modes, and where we have temporarily dropped the forcing term for conciseness. Note that because we are using the non-symmetric operator N instead of M, the cross-term which before was prefixed by a factor 2, is now split into two separate cross-terms. The significance of this will be seen when we analyse the spectral energy flux contributions for each of these terms.

From the above partitioned dynamical equations, the corresponding partitioned spectral energy equations may be constructed. To facilitate this we will use the following shorthand convention

$$\begin{aligned} T^{-+-}(k,t) &\equiv 2\pi k^2 N_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j \left\{ \left\langle u_\alpha^-(-\mathbf{k},t)\, u_\beta^+(\mathbf{j},t) u_\gamma^-(\mathbf{k}-\mathbf{j},t) \right\rangle - \left\langle u_\alpha^-(\mathbf{k},t)\, u_\beta^+(-\mathbf{j},t) u_\gamma^-(-\mathbf{k}+\mathbf{j},t) \right\rangle \right\} \\ &= 2\pi k^2 N_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j \left\{ \left\langle u_\alpha^-(-\mathbf{k},t)\, u_\beta^+(\mathbf{j},t) u_\gamma^-(\mathbf{k}-\mathbf{j},t) \right\rangle + c.c. \right\} , \end{aligned} \quad (5.13)$$

where $c.c.$ implies the complex conjugate of the term preceding it, and where the example of $T^{-+-}(k,t)$ is given. The $T^{-+-}(k,t)$ term represents the average energy transferred to or from a velocity mode with wavevector $\mathbf{k}$ in the resolved (low-k) range, due to all interactions between a resolved wavevector mode and a subgrid (high-k) wavevector mode. Using this convention we may write the low-k spectral energy equation as

$$\left[\frac{\partial}{\partial t} + 2\nu k^2\right] E^-(k,t) = T^{---}(k,t) + T^{-+-}(k,t) + T^{--+}(k,t) + T^{-++}(k,t) , \quad (5.14)$$

and the high-k spectral energy equation as

$$\left[\frac{\partial}{\partial t} + 2\nu k^2\right] E^+(k,t) \;=\; T^{+--}(k,t) + T^{++-}(k,t) + T^{+-+}(k,t) + T^{+++}(k,t)\;. \quad (5.15)$$

Note that it is important to remember the convention outlined in (5.13), as future results will rely on whether the wavevectors in question lie in the same range (denoted by '$+$' or '$-$').

5.2.1 Partitioned energy flux

5.2.1.1 Low-k energy flux

When equation (5.14) is integrated over all k-space, two terms vanish and two terms do not. These are summarised below

$$\int dk T^{---}(k,t) = 0\;, \qquad \int dk T^{--+}(k,t) = 0\;, \quad (5.16)$$

$$\int dk T^{-+-}(k,t) \neq 0\;, \qquad \int dk T^{-++}(k,t) \neq 0\;, \quad (5.17)$$

The proof for the relations in (5.16) is similar to the proof for relation (5.9); remembering that for the integral to vanish we require the integration limits of the k and j wavenumbers to be the same. In our convention (definition (5.13)) this corresponds to all terms with the first two signs in the superscript of the $T^{\square\square\square}(k,t)$ term being the same, irrespective of what the third sign is. With this shown, the low-k energy flux equation is

$$\int \frac{\partial}{\partial t} E^-(k,t)dk + \int 2\nu k^2 E^-(k,t)dk = \int dk\; \{\underbrace{T^{-+-}(k,t)}_{cross} + \underbrace{T^{-++}(k,t)}_{reynolds}\}\;, \quad (5.18)$$

where we have explicitly labeled the cross and Reynolds-like interaction terms.

5.2.1.2 High-k energy flux

In a similar way to the above analysis, when equation (5.15) for the high-k spectral energy is integrated over all k-space we obtain the results

$$\int dk T^{+++}(k,t) = 0 \ , \qquad \int dk T^{++-}(k,t) = 0 \ , \tag{5.19}$$

$$\int dk T^{+-+}(k,t) \neq 0 \ , \qquad \int dk T^{+--}(k,t) \neq 0 \ , \tag{5.20}$$

and the high-k energy flux equation is given by

$$\int \frac{\partial}{\partial t} E^{+}(k,t) dk + \int 2\nu k^2 E^{+}(k,t) dk = \int dk \ \{ \underbrace{T^{+-+}(k,t)}_{cross} + \underbrace{T^{+--}(k,t)}_{reynolds} \} \ , \tag{5.21}$$

where we have again labeled the equivalent cross and Reynolds like terms in the high-k case.

5.2.2 Energy conservation revisited

If we now add the two contributions to the total energy flux i.e. equations (5.18) and (5.21), we get

$$\begin{aligned} \int \left\{ \frac{\partial}{\partial t} E(k,t) + 2\nu k^2 E(k,t) \right\} dk \ &= \ \int dk \left\{ T^{-+-}(k,t) + T^{-++}(k,t) \right. \\ &\quad \left. + T^{+-+}(k,t) + T^{+--}(k,t) \right\} \\ &\quad + \int dk W(k,t) \ , \end{aligned} \tag{5.22}$$

where we have also returned the effects of the forcing term. For consistency with equation (5.8), the sum of all the non-linear inertial transfer terms on the RHS of equation (5.22) should vanish. This can be seen by realising that

$$\int dk T^{-+-}(k,t) \ = \ - \int dk T^{+--}(k,t) \ , \tag{5.23}$$

$$\int dk T^{-++}(k,t) \ = \ - \int dk T^{+-+}(k,t) \ , \tag{5.24}$$

thus confirming energy conservation in the non-linear term shown earlier in equation (5.9). The first of these identities can be proven by using

$$\int dkT^{\cdot\cdot-}(k,t) = 0 \ , \tag{5.25}$$

where '$\cdot$' in the superscript signifies no sign on the respective mode i.e. the mode is defined over all k-space. Remember that this relation holds true because the first two signs in the superscript of the inertial transfer term are the same (in this case they are both '$\cdot$'). By expanding the first sign in the superscript of (5.25)

$$\int dkT^{-\cdot-}(k,t) + \int dkT^{+\cdot-}(k,t) = 0 \ , \tag{5.26}$$

we obtain the relation

$$\int dkT^{-\cdot-}(k,t) = -\int dkT^{+\cdot-}(k,t) \tag{5.27}$$

The proof for identity (5.23) then follows by expanding the remaining second/middle sign in the superscripts of the integrands in (5.27)

$$\int dkT^{-+-}(k,t) + \int dkT^{---}(k,t) = -\int dkT^{++-}(k,t) - \int dkT^{+--}(k,t) \ , \tag{5.28}$$

and using $\int dkT^{---}(k,t) = 0$ and $\int dkT^{++-}(k,t) = 0$ from (5.16) and (5.19), the desired result is obtained. The proof for identity (5.24) follows in a similar fashion.

5.3 Numerical studies

It is quite clear that successful results in any coarse-grained model, LES or RG, depend upon how one deals with the cross and Reynolds terms and also whether one term is more important to model than the other. Any insight on the dynamical and (especially) the kinetic properties of these terms would help in their correct modeling. It is precisely this motivation that has instigated many researchers to conduct high-powered numerical studies in the from of a DNS to ascertain such information. In particular as discussed in Chapter 2, pragmatic needs dictate that the kinetics and thus the effects of the last three terms of the RHS of equation (5.14) for the low-k spectral energy, are worthy of the more

urgent study. In an LES model, the effects of these terms are responsible for the enhanced viscosity. DNS studies look at the form that this enhanced viscosity must take by constructing it from the raw $T(k,t)$ term obtained from the data. This augmentation, known as the *eddy viscosity*, to the 'bare' viscosity is given by

$$\begin{aligned} \delta\nu(k) &= -\frac{T^{-+-}(k,t)+T^{--+}(k,t)+T^{-++}(k,t)}{2k^2E^-(k,t)} \\ &= \delta\nu^{cross}(k)+\delta\nu^{Reyn}(k)\ , \end{aligned} \tag{5.29}$$

where the $\delta\nu^{cross}(k)$ comprises the contributions from both the T^{-+-} and T^{--+} terms.

We will start by a brief look at some semi-analytical methods that utilise statistical closure theories of the RPT form as being representative of the NSE, to construct terms like (5.29). The reason why we discuss these models first rather than jumping straight into DNS studies is because

1. semi-analytical models pre-date DNS studies and thus chronologically set the trend and formalism for studying eddy viscosities,
2. DNS studies, although run on the latest powerful computers, are only now touching Reynolds numbers of real interest with appreciable inertial ranges where scale-free behaviour is believed to dominate.

For details of all the statistical closures mentioned below, we refer the reader to Chapter 7. The discussion of semi-analytical models is followed by looking at DNS studies for the quantity in (5.29) and the relative importance of the cross and Reynolds terms in the kinetics of turbulent flow.

5.3.1 Semi-analytical studies

Most studies in this area were motivated by the work of Kraichnan [82, 83, 67] who used a single time (almost) Markovianized model known as the 'test-field model' (TFM), based upon his pioneering second-order DIA closure. Kraichnan studied the form that the eddy viscosity in relation (5.29) takes if we use the form of $T(k,t)$ from TFM, for the case of two and three dimensional isotropic turbulence. Assuming a spectral gap between large and small wavenumbers,

Kraichnan showed that for $k \ll k_c$ the eddy viscosity exhibits a constant asymptote/plateau given by

$$\delta\nu(k_c) = \frac{2\pi}{15}\int_{k_c}^{\infty} \theta_{qq0}[7Q(p) + pdQ(p)/dp]p^2 dp \ , \tag{5.30}$$

where θ_{qq0} is a characteristic interaction/memory time. If k_c is taken to be in the inertial range we find that equation (5.30) becomes

$$\delta\nu(k_c) = \frac{1}{12}\alpha^{1/2}\mu^{-1}\varepsilon^{1/3}k_c^{-4/3} \ , \tag{5.31}$$

for $k \ll k_c$, and where α is the Kolmogorov constant and μ is another constant of order unity. For the overall k-dependent eddy viscosity, relation (5.29) was calculated numerically. This showed that the the constant plateau behaviour that persists in the low-k region sharply turns up and exhibits a cusp-like behaviour near the partition k_c. An important point that Kraichnan makes is that at low-k the behaviour of the eddy viscosity is independent of the form of the energy spectrum $E(k)$. Near $k = k_c$, however, where the cusp behaviour dominates, the intensity of the cusp is dependent upon the form of $E(k)$; and in particular, is dependent upon the low-k energy spectrum behaviour i.e. the $\delta\nu(k)\ \lim k \to k_c$ behaviour depends upon the $E(k)\ \lim k \to 0$ behaviour. This point will be useful later on, when we discuss the modeling of the cross-term in our RG scheme. A last point regarding the TFM, is that it exhibits a divergence in the eddy viscosity at $k = k_c$ if the Kolmogorov $k^{-5/3}$ form for the energy spectrum is assumed to continue to $k = 0$. Kraichnan controls this by introducing a parameter analogous to an IR cut-off. The introduction and sensitivity to this cut-off is where the above realisation is made that although the plateau behaviour at $k \ll k_c$ is independent of $E(k)$, the cusp behaviour near $k \sim k_c$ is not.

Chollet and Lesieur [84, 85, 86] have proposed a similar technique to the one by Kraichnan of obtaining an eddy viscosity with a spectral gap valid in the $k \ll k_c$ area, but using the 'eddy damped quasi-normal Markovian' (EDQNM) model of Orszag [87] instead. They obtained an expression similar to equation (5.30) for $\delta\nu(k_c)$ which when evaluated assuming k_c lies within the Kolmogorov inertial range, gives

$$\delta\nu(k_c) = 0.28\left[\frac{E(k_c)}{k_c}\right]^{1/2} \ , \tag{5.32}$$

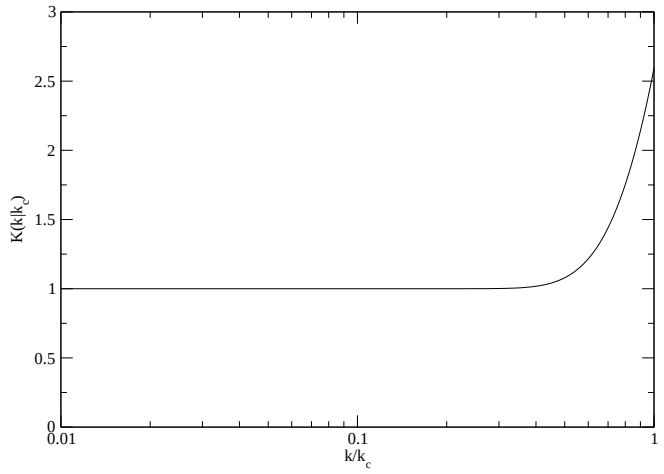

Figure 5.2: The form of the k-dependent spectral eddy viscosity as shown by the function $K(k|k_c)$ in EDQNM.

where the 0.28 factor comes from assuming $\alpha = 1.4$. The k dependence is then modeled by the introduction of a function $K(k|k_c)$ (first introduced by Kraichnan [67] in the above mentioned analysis), which is bolted onto equation (5.32) in the following way

$$\begin{aligned} \delta\nu(k|k_c) &= K(k|k_c)\delta\nu(k_c) \\ &= \left[1 + 32.9e^{-3(k_c/k)}\right]\delta\nu(k_c)\ , \end{aligned} \tag{5.33}$$

and where the second line defines $K(k|k_c)$. The form of this function is approximately constant and equal to one, except near $k \sim k_c$ where it exhibits the cusp behaviour as desired. The form of the k-dependent eddy viscosity is shown in figure 5.2 where the function $K(k|k_c)$ is plotted.

Schilling and Zhou [88] did a similar study as Kraichnan but with EDQNM i.e. they have used $T(k,t)$ from EDQNM and used a similar relation to (5.29) in order to obtain the eddy viscosity. In particular they have split the $T(k,t)$ term into two parts; one which they call the 'eddy viscosity' and the other which they call the 'backscatter viscosity'[1]. We define our eddy viscosity differently

[1]As discussed in Chapter 7, this splitting of the $T(k,t)$ into two parts and calling one the eddy viscosity, has been pointed out to be essentially arbitrary by McComb, and led the latter to form a variation of the Edwards-Fokker-Planck closure for stationary turbulence.

from Schilling and Zhou, as the sum of their eddy and backscatter viscosities. Furthermore, they have numerically calculated the contribution of the Reynolds and cross term to each of these viscosities. Without going into too much detail (as we are not concerned with separate eddy and backscatter viscosities but the sum of both), one of the principal results of their study is that the main contributions from the Reynolds and cross-components of the eddy viscosity arise from modes with $k/k_c \ll 1$ and $k/k_c \lesssim 1$ respectively.

The recent study of Nakano, McComb and Guerts [89] is a similar study to Schilling and Zhou but using the Edwards-Fokker-Planck [90] closure model for stationary turbulence. Their study not only indicated similar results to the studies above, but also clearly showed that by far the major contribution to the eddy viscosity plateau is from the Reynolds term alone, and the major contribution to the cusp is from the cross term. They also study the form of the transfer spectra for the Reynolds term and both of the non-symmetric contributions to the cross-term. In particular they showed, both analytically and numerically, the energy flux results obtained in relations (5.16) and (5.19), and also showed that if we have k_c in the inertial range, then the ratio of the energy fluxes from the cross and Reynolds term is

$$\frac{\int dk T^{-+-}(k,t)}{\int dk T^{-++}(k,t)} = \frac{cross}{Reynolds} \approx 0.85 \ , \tag{5.34}$$

i.e. in the inertial range, the fluxes from the cross and Reynolds terms are comparable but the Reynolds term is still the larger. Lastly, it is worth mentioning that this study also introduces an interesting method of including the effects of the cross-term in subgrid models by the introduction of slaving a component of the cross term subgrid scales to the resolved scales in an LES type model.

5.3.2 DNS studies

Because, advances in computation are rapid, we will start with the oldest relevant DNS based studies and finish with the newest ones.

The DNS study (64^3 and 128^3 grid points; decaying turbulence) of Domaradzki *et. al.* [91] showed that although they obtain the familiar cusp behaviour near $k \sim k_c$, their results for $k \leq 0.5k_c$ show a *negative* behaviour in the eddy viscosity. They attribute this to the lack of an appreciable inertial range for the

low Reynolds number simulations that they conducted.

The DNS study (128^3 grid points; decaying turbulence) that Zhou and Vahala included with the paper on their RG scheme [65] re-iterated clearly that the cross and Reynolds interaction terms are responsible for the cusp and plateau behaviour in the eddy viscosities respectively. This supported their argument that the cross interaction in terms of a triple non-linearity should be retained recursively in any RG calculation. This was backed up later in an informative paper by Dubois, Jauberteau and Zhou [80] (128^3 grid points; stationary turbulence $Re_\lambda \sim 70$) who conducted their study not in the context of an eddy viscosity but by investigating the kinetic effects that are caused by neglecting each of the cross and Reynolds terms in turn. They studied the energy and dissipation spectra and showed that keeping the cross term and neglecting the Reynolds term showed very good agreement with DNS, but vice versa was not satisfactory.

The DNS study of McComb and Young [92, 93] (256^3 grid points; stationary turbulence $Re_\lambda \sim 190$) has provided a thorough investigation into the effects of the cross and Reynolds terms by looking at both eddy viscosities as well as partitioned spectra. Of particular interest in the latter study was the investigation into the effects of varying the partitioning wavenumber k_c on the transfer spectra and flux. They showed that varying k_c changed the ratio of the energy flux contributions from the cross and Reynolds term. If k_c is in the far dissipation region $k_c > k_d$ then the flux contribution from the cross term is far higher than the Reynolds term contribution; whereas if k_c moves towards the inertial Kolmogorov scaling range $k_c < k_d$ then the contribution from the Reynolds term becomes more comparable with the cross term. This latter result agrees with the study of Nakano, McComb and Guerts mentioned earlier. A recent study by Kuczaj and Guerts [94] has taken this study further by scanning k_c through all k-space. Their results support the results of McComb and Young but also show that as k_c passes through the inertial and into the forcing range we see that a swap occurs and the Reynolds term flux contributions become larger than the cross term contributions.

Lastly, the very recent results of Hughes, Wells and Wray [95] (160^3 grid points; stationary turbulence) support all the above results and also highlight the result that the effect of the cusp is prominent only for k_c in high-k dissipative range.

These are interesting results and have important implications for all sub-grid

term	contributes to	contains terms like	x-space counterpart
$T^{-+-}(k)$	low-k energy flux	$k_\gamma u_\beta^+(\mathbf{j}) u_\gamma^-(\mathbf{k}-\mathbf{j})$	$u_\gamma^-(\mathbf{x})\partial_\gamma u_\beta^+(\mathbf{x})$
$T^{-++}(k)$	low-k energy flux	$k_\gamma u_\beta^+(\mathbf{j}) u_\gamma^+(\mathbf{k}-\mathbf{j})$	$u_\gamma^+(\mathbf{x})\partial_\gamma u_\beta^+(\mathbf{x})$
$T^{+-+}(k)$	high-k energy flux	$k_\gamma u_\beta^-(\mathbf{j}) u_\gamma^+(\mathbf{k}-\mathbf{j})$	$u_\gamma^+(\mathbf{x})\partial_\gamma u_\beta^-(\mathbf{x})$
$T^{+--}(k)$	high-k energy flux	$k_\gamma u_\beta^-(\mathbf{j}) u_\gamma^-(\mathbf{k}-\mathbf{j})$	$u_\gamma^-(\mathbf{x})\partial_\gamma u_\beta^-(\mathbf{x})$

Table 5.1: Table showing the dynamical interactions responsible for non-vanishing energy flux terms and their corresponding x-space counterpart terms.

models. To summarise we see that:

- the cross-term is by far the most important energy flux contributor and swamps out the Reynolds-term in the far dissipation range where $k \sim k_d$; at and near the inertial range the two terms flux contributions are comparable; and preliminary results suggest that in the large-scale forcing range the Reynolds-term becomes the major contributor to energy flux;
- the cross-term is responsible for the cusp behaviour of the eddy viscosity and the Reynolds-term is responsible for the constant plateau behaviour.
- the plateau feature seems to have a universal form, whilst the cusp is dependent upon the large scale energy spectrum behaviour.

Finally, and as a point of interest, by looking at our high and low-k energy flux expressions (5.18) and (5.21), and remembering that the pressure does not contribute to energy transfer, one can make a x-space comparison of the important convective terms. These are shown in table 5.1 .

5.3.3 Implications for RG models

The above results quite clearly show that the modeling of the cross term is very much the important factor for a coarse-grain model with the k_c partitioning wavenumber in the dissipative range. The previous RG analysis of McComb and Johnston [64] has an error in a calculation which caused them to neglect the cross term on the grounds that it does not contribute to the energy flux. This error arose due to the mixup between the two cross-terms in (5.11). This in turn is most probably due to the use of the symmetric $M_{\alpha\beta\gamma}(\mathbf{k})$ operator rather

than the non-symmetric $N_{\alpha\beta\gamma}(\mathbf{k})$ operator that we have chosen to use so as not to confuse the different cross-term effects. This being so we are now faced with the task of including the effects of the cross-term in our RG calculation. This will also allows us to respond to the consistency criticism mentioned in the previous chapter, that unlike the Reynolds-term, the cross-term interaction was not expanded out to second order in the local Reynolds number. The next section describes a simple model in order to heuristically include the effects of the cross-term. Note that for the following analysis we will not start with a dimensionless equation like in Chapter 4. We will also not do any rescaling analysis, as it is essentially identical to the previous analysis. However, we will keep track of terms by introduction of a book-keeping parameter λ_0 which would be equivalent of the local Reynolds number in the dimensionless NSE case.

5.4 Including the cross-term interaction in the RG coarse-graining

From the low-k NSE as given by (5.11) we construct an equation similar to equation (1.29) which is simply an intermediate equation before the low-k energy equation (5.14)

$$\begin{aligned}\left[\frac{\partial}{\partial t}+2\nu_0 k^2\right]\left\langle u_\alpha^-(-\mathbf{k},t)u_\alpha^-(\mathbf{k},t)\right\rangle &= \frac{1}{2}\left[N^-_{\alpha\beta\gamma}(\mathbf{k})+N^-_{\alpha\gamma\beta}(\mathbf{k})\right]\lambda_0\int d^3j\times\\ &\times\left\{2\left(\left\langle u_\alpha^-\left(-\mathbf{k},t\right)u_\beta^+(\mathbf{j},t)u_\gamma^-(\mathbf{k}-\mathbf{j},t)\right\rangle-c.c.\right)\right.\\ &+\left(\left\langle u_\alpha^-\left(-\mathbf{k},t\right)u_\beta^+(\mathbf{j},t)u_\gamma^+(\mathbf{k}-\mathbf{j},t)\right\rangle-c.c.\right)\\ &\left.+\left(\left\langle u^-u^-u^-\right\rangle-c.c.\right)\right\}\ , \end{aligned}\tag{5.35}$$

where $c.c.$ implies the complex conjugate of the term before it, λ_0 is a book-keeping parameter eventually set to unity, and we have put the $\langle u^-u^-u^-\rangle$ in shorthand notation because it does not require any further coarse-graining. As before, we have made the implicit assumption of a forcing on the RHS of equation (5.35). We have also simplified matters greatly by rewriting dummy indices and variables to the extent that we can replace all the N operators in equation (5.35), with M operators by using equation (5.4). This will not affect any future calculation as we have already ascertained the importance of the

cross-term in the earlier sections.

Let us now look at the ensemble average in equation (5.35). In the earlier RG studies of McComb *et. al.*, and especially in the latest work [64], it is discussed that 'the full ensemble average, needed to establish the energy balance, relaxes the constraint on the conditional average' [96]. In fact it can be shown that

$$\langle A \rangle \equiv \left\langle \hat{\mathcal{P}}_{\mathcal{C}} \left\{ A \right\} \right\rangle \ , \tag{5.36}$$

for some random variable A i.e. that a full average over an ensemble comprised of individual realisations of A is equivalent to a full average over an ensemble which comprises all the different conditional averages of A. In the former all the individuals realisations are being included in the average, whilst in the latter it is the different conditions (sub-ensembles) which are being averaged. In either case the total sets comprise of the same realisations. We will now use this useful equivalence to handle the coarse-graining of equation (5.35).

We write equation (5.35) using the equivalence in (5.36), and seeing that we can pull the ensemble average angle brackets out so as to be at the peripheries of the written equation e.g. we can write the LHS of equation (5.35) as

$$\left[\frac{\partial}{\partial t} + 2\nu_0 k^2 \right] \left\langle u_\alpha^-(-\mathbf{k},t) u_\alpha^-(\mathbf{k},t) \right\rangle \rightarrow \left\langle \left[\frac{\partial}{\partial t} + 2\nu_0 k^2 \right] \hat{\mathcal{P}}_{\mathcal{C}} \left\{ u_\alpha^-(-\mathbf{k},t) u_\alpha^-(\mathbf{k},t) \right\} \right\rangle \ , \tag{5.37}$$

we will assume the existence of the peripheral angle brackets implicitly. This being so, we have equation (5.35) as

$$\begin{aligned} \left[\frac{\partial}{\partial t} + 2\nu_0 k^2 \right] \hat{\mathcal{P}}_{\mathcal{C}} \left\{ u_\alpha^-(-\mathbf{k},t) u_\alpha^-(\mathbf{k},t) \right\} \ &= \ \lambda_0 M^-_{\alpha\beta\gamma}(\mathbf{k}) \int d^3 j \times \\ &\times \Big\{ 2 \left(\hat{\mathcal{P}}_{\mathcal{C}} \left\{ u_\alpha^- \left(-\mathbf{k},t\right) u_\beta^+(\mathbf{j},t) u_\gamma^-(\mathbf{k}-\mathbf{j},t) \right\} - c.c. \right) \\ &+ \left(\hat{\mathcal{P}}_{\mathcal{C}} \left\{ u_\alpha^- \left(-\mathbf{k},t\right) u_\beta^+(\mathbf{j},t) u_\gamma^+(\mathbf{k}-\mathbf{j},t) \right\} - c.c. \right) \\ &+ \left(\hat{\mathcal{P}}_{\mathcal{C}} \left\{ u^- u^- u^- \right\} - c.c. \right) \Big\} \ , \end{aligned} \tag{5.38}$$

which when the CA is evaluated, becomes

$$\begin{aligned}\left[\frac{\partial}{\partial t}+2\nu_0 k^2\right]u_\alpha^-(-\mathbf{k},t)u_\alpha^-(\mathbf{k},t) &= \lambda_0 M^-_{\alpha\beta\gamma}(\mathbf{k})\int d^3j \times \\ &\times\Big\{2\left(u_\alpha^-(-\mathbf{k},t)\,u_\gamma^-(\mathbf{k}-\mathbf{j},t)\hat{\mathcal{P}}_{\mathcal{C}}\left\{u_\beta^+(\mathbf{j},t)\right\}-c.c.\right) \\ &+\left(u_\alpha^-(-\mathbf{k},t)\,\hat{\mathcal{P}}_{\mathcal{C}}\left\{u_\beta^+(\mathbf{j},t)u_\gamma^+(\mathbf{k}-\mathbf{j},t)\right\}-c.c.\right) \\ &+\left(u^-u^-u^--c.c.\right)\Big\}\ . \end{aligned} \tag{5.39}$$

It is easily seen that the Reynolds term analysis is essentially the same as our previous RG analysis in Chapter 4; so we will just use the results for this as the Reynolds term contribution to the increment to the viscosity.

We now handle the cross-term. Writing a dynamical equation for $u_\beta^+(\mathbf{j},t)$ using equation (5.12), we get

$$\begin{aligned}\left[\frac{\partial}{\partial t}+\nu_0 j^2\right]u_\beta^+(\mathbf{j},t) &= \lambda_0 M^+_{\beta\rho\sigma}(\mathbf{j})\int d^3p\left\{u_\rho^-(\mathbf{p},t)u_\sigma^-(\mathbf{j}-\mathbf{p},t)\right. \\ &\left.+2u_\rho^-(\mathbf{p},t)u_\sigma^+(\mathbf{j}-\mathbf{p},t)+u_\rho^+(\mathbf{p},t)u_\sigma^+(\mathbf{j}-\mathbf{p},t)\right\}\ . \end{aligned} \tag{5.40}$$

Inverting the linear operator on the LHS and applying the CA, we obtain

$$\begin{aligned}\hat{\mathcal{P}}_{\mathcal{C}}\left\{u_\beta^+(\mathbf{j},t)\right\} &= \lambda_0 M^+_{\beta\rho\sigma}(\mathbf{j})\int_{-\infty}^{t} ds e^{-\nu_0 j^2(t-s)}\int d^3p\left\{u_\rho^-(\mathbf{p},s)u_\sigma^-(\mathbf{j}-\mathbf{p},s)\right. \\ &\left.+2u_\rho^-(\mathbf{p},s)\hat{\mathcal{P}}_{\mathcal{C}}\left\{u_\sigma^+(\mathbf{j}-\mathbf{p},s)\right\}+\hat{\mathcal{P}}_{\mathcal{C}}\left\{u_\rho^+(\mathbf{p},s)u_\sigma^+(\mathbf{j}-\mathbf{p},s)\right\}\right\}\ . \end{aligned} \tag{5.41}$$

Before we substitute this back into equation (5.39), let us save some space by noting that to handle the last two terms on the RHS of equation (5.41) we will need to construct further dynamical equations and thus will introduce higher order terms in the local Reynolds number. Because we are only working to second order, we will not include these extra terms in the analysis[2]. Thus,

[2]We will come back to this point later in the Conclusion (Chapter 9).

substituting (5.41) into equation (5.39) and focusing on the cross term we get

$$\begin{aligned} cross-term &= \lambda_0^2 \int d^3j 2M^-_{\alpha\beta\gamma}(\mathbf{k})M^+_{\beta\rho\sigma}(\mathbf{j}) \int_{-\infty}^{t} ds e^{-\nu_0 j^2(t-s)} \int d^3p \times \\ &\quad \times \left[\left\langle u^-_\alpha\left(-\mathbf{k},s\right) u^-_\gamma(\mathbf{k}-\mathbf{j},s) u^-_\rho(\mathbf{p},s) u^-_\sigma(\mathbf{j}-\mathbf{p},s)\right\rangle - c.c.\right] , \end{aligned} \tag{5.42}$$

where we have also brought back in the peripheral ensemble average brackets to show the next simplification; this being that as we are only considering statistically stationary systems the time-history integral does not affect the quartic moment in the integrand. This being so, we can simplify equation (5.42) to the form

$$\begin{aligned} cross-term &= \lambda_0^2 \int d^3j \frac{2M^-_{\alpha\beta\gamma}(\mathbf{k})M^+_{\beta\rho\sigma}(\mathbf{j})}{\nu_0 j^2} \int d^3p \times \\ &\quad \times \left[\left\langle u^-_\alpha\left(-\mathbf{k},t\right) u^-_\gamma(\mathbf{k}-\mathbf{j},t) u^-_\rho(\mathbf{p},t) u^-_\sigma(\mathbf{j}-\mathbf{p},t)\right\rangle - c.c.\right] , \end{aligned} \tag{5.43}$$

where the time-history integral has resulted in a $\nu_0 j^2$ 'lifetime' factor in the denominator. To handle the quartic non-linearity (which is a cubic non-linearity in the dynamics), we employ a crude 'quasi-normal' trick used by Zhou *et. al.* which assumes that the fourth order moment can be factored as if it was a Gaussian. This implies that

$$\begin{aligned} &\left\langle u^-_\alpha\left(-\mathbf{k},t\right) u^-_\gamma(\mathbf{k}-\mathbf{j},t) u^-_\rho(\mathbf{p},t) u^-_\sigma(\mathbf{j}-\mathbf{p},t)\right\rangle \\ &\quad = \left\langle u^-_\alpha\left(-\mathbf{k},t\right) u^-_\gamma(\mathbf{k}-\mathbf{j},t)\right\rangle \left\langle u^-_\rho(\mathbf{p},t) u^-_\sigma(\mathbf{j}-\mathbf{p},t)\right\rangle \\ &\quad + \left\langle u^-_\alpha\left(-\mathbf{k},t\right) u^-_\rho(\mathbf{p},t)\right\rangle \left\langle u^-_\gamma(\mathbf{k}-\mathbf{j},t) u^-_\sigma(\mathbf{j}-\mathbf{p},t)\right\rangle \\ &\quad + \left\langle u^-_\alpha\left(-\mathbf{k},t\right) u^-_\sigma(\mathbf{j}-\mathbf{p},t)\right\rangle \left\langle u^-_\gamma(\mathbf{k}-\mathbf{j},t) u^-_\rho(\mathbf{p},t)\right\rangle , \end{aligned} \tag{5.44}$$

and is a result of what is sometimes known as Wick's Theorem [3]. We now apply relation (1.20) to all the 2^{nd} terms in the product i.e. if we have $\langle A\rangle \langle B\rangle$, then we apply (1.20) to $\langle B\rangle$. This being done we obtain

$$\begin{aligned} &\left\langle u^-_\alpha\left(-\mathbf{k},t\right) u^-_\gamma(\mathbf{k}-\mathbf{j},t) u^-_\rho(\mathbf{p},t) u^-_\sigma(\mathbf{j}-\mathbf{p},t)\right\rangle \\ &\quad = \left\langle u^-_\alpha\left(-\mathbf{k},t\right) u^-_\gamma(\mathbf{k}-\mathbf{j},t)\right\rangle \delta(\mathbf{j}) P_{\gamma\sigma}(\mathbf{p}) Q^-(p) \\ &\quad + \left\langle u^-_\alpha\left(-\mathbf{k},t\right) u^-_\rho(\mathbf{p},t)\right\rangle \delta(\mathbf{k}-\mathbf{p}) P_{\gamma\sigma}(\mathbf{k}-\mathbf{j}) Q^-(|\mathbf{k}-\mathbf{j}|) \\ &\quad + \left\langle u^-_\alpha\left(-\mathbf{k},t\right) u^-_\sigma(\mathbf{j}-\mathbf{p},t)\right\rangle \delta(\mathbf{k}-\mathbf{j}+\mathbf{p}) P_{\gamma\rho}(\mathbf{k}-\mathbf{j}) Q^-(|\mathbf{k}-\mathbf{j}|) , \end{aligned} \tag{5.45}$$

where all time arguments have been ignored as we are assuming statistical stationarity as before. Now substituting this result back into equation (5.42), we find that the first term in (5.45) will give zero as the $M^{+}_{\beta\rho\sigma}(\mathbf{j})$ is in the high-k band whereas the $\delta(\mathbf{j})$ demands that $\mathbf{j} = 0$. Performing the p integration on the rest of the terms gives

$$\begin{aligned} cross-term &= \lambda_0^2 \int d^3j \frac{2M^{-}_{\alpha\beta\gamma}(\mathbf{k})M^{+}_{\beta\rho\sigma}(\mathbf{j})}{\nu_0 j^2} \times \\ &\quad \times \left\{ \left\langle u^{-}_{\alpha}(-\mathbf{k},t)\, u^{-}_{\rho}(\mathbf{k},t) \right\rangle P_{\gamma\sigma}(\mathbf{k}-\mathbf{j}) Q^{-}(|\mathbf{k}-\mathbf{j}|) \right. \\ &\quad \left. + \left\langle u^{-}_{\alpha}(-\mathbf{k},t)\, u^{-}_{\sigma}(\mathbf{k},t) \right\rangle P_{\gamma\rho}(\mathbf{k}-\mathbf{j}) Q^{-}(|\mathbf{k}-\mathbf{j}|) \right\} \\ &\quad - c.c. \; . \end{aligned} \tag{5.46}$$

Up to this point we have not mentioned any analysis of the $c.c.$ term. After a similar analysis is done for these terms, we can write equation (5.46) as

$$\begin{aligned} cross-term &= \lambda_0^2 \int d^3j \frac{2Q^{-}(|\mathbf{k}-\mathbf{j}|)}{\nu_0 j^2} \times \\ &\quad \times \left\{ \left\langle u^{-}_{\alpha}(-\mathbf{k},t)\, u^{-}_{\rho}(\mathbf{k},t) \right\rangle \underline{M^{-}_{\alpha\beta\gamma}(\mathbf{k})M^{+}_{\beta\rho\sigma}(\mathbf{j})P_{\gamma\sigma}(\mathbf{k}-\mathbf{j})} \right. \\ &\quad + \left\langle u^{-}_{\alpha}(-\mathbf{k},t)\, u^{-}_{\sigma}(\mathbf{k},t) \right\rangle \underline{M^{-}_{\alpha\beta\gamma}(\mathbf{k})M^{+}_{\beta\rho\sigma}(\mathbf{j})P_{\gamma\rho}(\mathbf{k}-\mathbf{j})} \\ &\quad + \left\langle u^{-}_{\alpha}(\mathbf{k},t)\, u^{-}_{\rho}(-\mathbf{k},t) \right\rangle \underline{M^{-}_{\alpha\beta\gamma}(\mathbf{k})M^{+}_{\beta\rho\sigma}(\mathbf{j})P_{\gamma\sigma}(\mathbf{k}-\mathbf{j})} \\ &\quad \left. + \left\langle u^{-}_{\alpha}(\mathbf{k},t)\, u^{-}_{\sigma}(-\mathbf{k},t) \right\rangle \underline{M^{-}_{\alpha\beta\gamma}(\mathbf{k})M^{+}_{\beta\rho\sigma}(\mathbf{j})P_{\gamma\rho}(\mathbf{k}-\mathbf{j})} \right\} \; . \end{aligned} \tag{5.47}$$

The underlined isotropic tensor terms in the above equation may be handled by relation (4.21). Once all the tensor contractions are done, the above equation may be simplified to the form

$$cross-term = -\lambda_0^2 \int d^3j \frac{4L(\mathbf{k},\mathbf{j}^{+})Q^{-}(|\mathbf{k}-\mathbf{j}|)}{\nu_0 j^2} \left\langle u^{-}_{\alpha}(\mathbf{k},t)\, u^{-}_{\sigma}(-\mathbf{k},t) \right\rangle \; , \tag{5.48}$$

where by looking at equation (5.35), we can see that the cross term is now in a form where it can be seen as an increment to the viscosity; and where the L coefficient is given by relation (4.23) and the argument has a '$+$' on the superscript of the $\mathbf{j}$ to remind us that $\mathbf{j}$ is in the high-k band. Substituting (5.48) back into equation (5.39) and writing this equation out in full (now renormalized)

form gives

$$\left[\frac{\partial}{\partial t} + 2\nu_1 k^2\right] \left\langle u_\alpha^-(-\mathbf{k},t) u_\alpha^-(\mathbf{k},t)\right\rangle$$
$$= \lambda_0 M^-_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j \left(\left\langle u_\alpha^-\left(-\mathbf{k},t\right) u_\beta^-(\mathbf{j},t) u_\gamma^-(\mathbf{k}-\mathbf{j},t)\right\rangle - c.c.\right), \quad (5.49)$$

where the angle brackets have been pulled back in, and where

$$\nu_1 = \nu_0 + \delta\nu_0^{Reyn} + \delta\nu_0^{cross}\ , \quad (5.50)$$

and

$$\delta\nu_0^{cross} = \frac{1}{k^2}\int d^3j \frac{2L(\mathbf{k},\mathbf{j}^+)Q^-(|\mathbf{k}-\mathbf{j}|)}{\nu_0 j^2}\ , \quad (5.51)$$

and earlier we mentioned that $\delta\nu_0^{Reyn}$, the Reynolds term viscosity increment, is the same as our older result

$$\delta\nu_0^{Reyn} = \frac{1}{k^2}\int d^3j \frac{L(\mathbf{k},\mathbf{j}^+)Q^+(|\mathbf{k}-\mathbf{j}|)}{\nu_0 j^2 + \nu_0|\mathbf{k}-\mathbf{j}|^2 - \nu_0 k^2}\ . \quad (5.52)$$

5.4.1 RG recursion equations

When equations (5.50)–(5.52) are written in terms of the spectral energy function $E(k)$, and made dimensionless to take account of the rescaling[3], we get the following final equations to facilitate the actual RG iteration

$$\nu'_{n+1}(k') \;=\; h^{4/3}\nu'_n(hk') + h^{-4/3}\delta\nu'_{n,Reyn}(k') + h^{-4/3}\delta\nu'_{n,cross}(k')\ , \quad (5.53)$$

$$\delta\nu'_{n,Reyn}(k') = \frac{1}{4\pi k'^2}\int d^3j' \frac{L(\mathbf{k}',\mathbf{j}')\left[h^{11/3} - (|\mathbf{k}'-\mathbf{j}'| - h^{-1})\,\frac{11}{3}h^{14/3}\right]}{\nu'_n(hj')j'^2 + \nu'_n(h\,|\mathbf{k}'-\mathbf{j}'|)\,|\mathbf{k}'-\mathbf{j}'|^2 - \nu'_n(hk')k'^2}\ , \quad (5.54)$$

$$\delta\nu'_{n,cross}(k') = \frac{1}{4\pi k'^2}\int d^3j' \frac{2L(\mathbf{k}',\mathbf{j}')\,|\mathbf{k}'-\mathbf{j}'|^{-11/3}}{\nu'_n(hj')j'^2}\ , \quad (5.55)$$

where $0 \le k' \le 1$ and $1 \le j', |\mathbf{k}'-\mathbf{j}'| \le h^{-1}$ for the Reynolds term, and $0 \le k', |\mathbf{k}'-\mathbf{j}'| \le 1$ and $1 \le j' \le h^{-1}$ for the cross term.

[3] All of which is identical to the analysis in Chapter 4.

5.4.1.1 Divergences

If we write equation (5.55) for the cross term increment to the viscosity, as

$$\delta\nu'_{n,cross}(k') = \frac{1}{4\pi k'^2}\int d^3j' \int d^3l' \delta(\mathbf{k}-\mathbf{j}-\mathbf{l})\frac{2L(\mathbf{k}',\mathbf{j}')l'^{-11/3}}{\nu'_n(hj')j'^2} \ , \tag{5.56}$$

we find that this exhibits a divergence as $l \to 0$ i.e. an IR divergence. This is primarily due to us taking the Kolmogorov spectrum to apply even at $k = 0$, which it only does hypothetically for the case of infinite Reynolds number. Zhou & Vahala [65], who also have similar problems, tame this divergence by using a more realistic (i.e. not infinite Reynolds number) energy spectrum which goes to zero at $k = 0$. They use a production type spectrum which is compatible with the Kolmogorov spectrum in a part of its range. This production spectrum was first introduced by Leslie and Quarini [66], and depends upon some parameters to fix the shape of the spectrum. In the current work we will instead tame our divergence by the crude but equivalent method of introducing an IR cut-off δ in the lower limit of the l integral i.e.

$$\int d^3l' = \int d\Omega_l \int_0^{k_n} dl' \to \int d\Omega_l \int_\delta^{k_n} dl' \ , \tag{5.57}$$

where we have transformed to spherical-polar coordinates, $d\Omega_l$ is the surface integral measure, and k_n is the n^{th} shell partition wavenumber. We will then investigate the sensitivity of this parameter δ on the resulting fixed point renormalized viscosity.

5.4.2 Results

The results for the fixed point renormalized viscosity in figure 5.3 show that the effects of including the cross term agree qualitatively with those presented in the numerical studies reviewed above. The renormalized viscosity exhibits a cusp as $k' \to 1$ which is equivalent to $k \to k_c$ in the unscaled case. Moreover, the results also support Kraichnan's comments that the form of the cusp is not a universal phenomenon, and is dependent upon the form of the energy spectrum. We can see this quite clearly as the intensity of the cusp becomes less prominent as one increase the IR limiting cut-off δ. We can also see that the renormalized viscosity also exhibits near constant behaviour at low

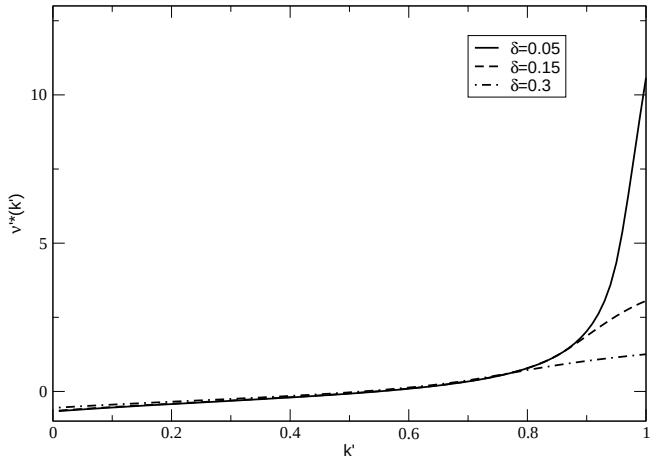

Figure 5.3: Scaled fixed point renormalized viscosities for shell width $\eta = 0.3$ at three values of the integral divergence cut-off δ. Note the dependence of the intensity of the cusp on δ and the independence to δ as $k' \to 0$.

k', and is independent of variations in δ. This also agrees with the comments of Kraichan and others. This independence to δ suggests that the constant behaviour at low k' arises mainly from the Reynolds term. This is due to the Reynolds term involving only integrals in the subgrid band and thus cannot be affected by any cut-offs in the low-k band; whereas on the other hand, as the cross term involves integrals in the low-k band, it is not surprising that it should be dependent upon δ and thus one could conclude that it is the key contributor to the cusp. This latter conclusion is also quite clear from the very fact that the cusp behaviour only decided to appear after we included the effects of the cross term.

We also notice that although nearly constant at low-k, the renormalized viscosity tends to become negative. This is most likely symptomatic of the use of the quasi-normality (QN) hypothesis to model the cross term. In Chapter 7 we review an early closure model based upon this hypothesis. The principal reason for the failure of this QN closure model was due to the unphysical occurrence of the energy spectrum becoming negative. Orszag's 1970 paper [87] attributed this to a build up of energy and suggested a cure via introducing an eddy damping parameter (see Chapter 7). With this is mind it is likely

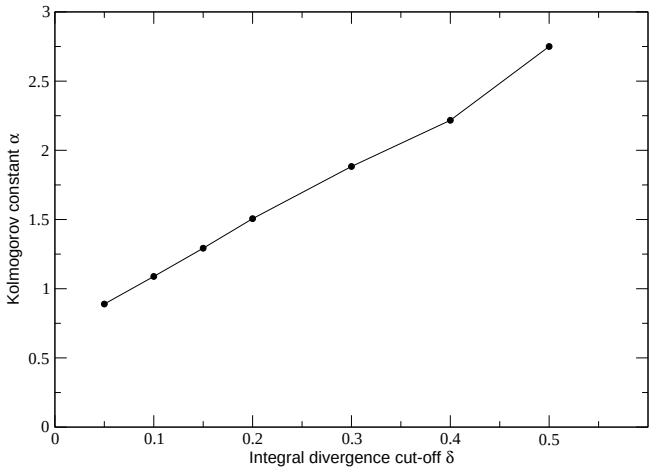

Figure 5.4: Graph showing the sensitivity of the calculated Kolmogorov constant to the IR cut-off δ, for $\eta = 0.3$.

that this build up is responsible for the negative viscosity, and thus there must be a part of the cross term which contributes to this low-k behaviour and not the Reynolds term as mentioned in the above paragraph. Clearly this possible build up is then responsible for the effects of the Reynolds term becoming insignificant as the RG iteration proceeds; thus showing that the cross-term must also have a δ independent part. However the value of the viscosity in this part of k-space is so small that when we look at the unscaled case, further discussion of cross-term contributions to this part seems pointless.

Lastly we also studied the sensitivity of the Kolmogorov constant α to the IR cut-off parameter δ, to see if α exhibits any constant behaviour i.e. the existence of a plateau which shows independence from δ , something to help us fix the value of δ. This notion was motivated by studies in other parts of Physics where artificial numerical parameters are chosen by minimising some key variable e.g. maximising the entropy or minimising the free-energy in statistical physics. However, the results for this study as illustrated in figure 5.4, also show a strong sensitivity to δ.

5.5 Comments

As the hypothesis of quasi-normality is such a gross assumption to make, we were only really expecting some qualitative indications that we are on the right track to account for the cusp like behaviour, and thus the 'hand-wavy' account of the cross-term and local interactions across the cut-off. The model presented in the previous section exhibits qualitatively correct results but the divergences are pathological. They arise essentially because we are bolting on an effect which is purely low-k in origin and thus should really be kept in the coarse-grained picture.

Although we had to refer to the energy picture to get the cross term contribution to the renormalized viscosity, this hypothesis of equating a kinetically based eddy viscosity to a dynamical one is not new; in fact nearly all LES models use this, including all the ones discussed in section 5.3 on numerical studies. To do otherwise would require an individual realisation based eddy viscosity, which is a formidable task as one would have to bring in phase effects etc.

One must remember that the cross-term and the proper accounting of its effects, is essentially the stumbling block on many RG schemes, as it involves interactions across the cut-off between the coarse-grained picture that we want and the subgrid picture that we are trying to average away. From the above comments, we anticipate that since the cross-term is essentially where the coarse-graining picture becomes one in which information on individual realisations becomes important (and thus phase effects), that a model which retains some kind of history is needed i.e. a Lagrangian type model. In this context it is worth mentioning that the method of retaining a triple non-linearity at every stage of the RG iteration, as in the work of Rose [58] and Zhou *et. al.* [65] seems like a way of retaining a history. This method of including cubic non-linearities has already been applied to the McComb RG scheme by Storkey [74] who essentially found similar results to Zhou *et. al.* with a divergence at $k \to k_c$. From looking at the results of Zhou *et. al.* it is also evident that including these cubic non-linearities at every stage in the RG analysis would also sort out the negative behaviour of the renormalized viscosity, as the QN hypothesis is only used after the RG iteration has been completed. One might also consider the use of a more realistic EDQN type hypothesis which includes the effects of eddy damping to sort out the latter problem.

Lastly, due to our preliminary results as well as the results of many other authors, it seems that the cross-term interaction mainly exhibits non-universal features. Although we have mentioned that the cross-term would probably contribute a little to the universal features of the fixed point renormalized viscosity, we anticipate that the majority of contribution comes from the Reynolds term. Thus our analysis of the previous chapter, which predicts a very good value of the *universal* Kolmogorov constant α, might still be valid. The partitioning of contributions is still a very much active area of research, as can be seen by the review of section 5.3; and thus also in the area of the applications of RG to these studies. Thus, in the absence of a properly comprehensive model, and in the hope that we are renormalizing the universal aspects of the viscosity in a sufficient enough way by the Reynolds-term only, the next chapter will be applying the RG analysis of Chapter 4 (without the cross-term) to the case of a passive scalar advected by a turbulent velocity field.

Chapter 6

Dynamic RG applied to the turbulent advection of a passive scalar

The application of the two-field theory RG, developed by McComb and Watt [62], to the advection of a passive scalar was first studied by Watt [73]. The two-field theory RG has since been updated. In particular the recent modifications by McComb & Johnston [64] have now taken into account effects which were previously approximated by a Markovianizing step in the time-integrals of the subgrid velocity modes equation. This removal of the Markovianizing approximation along with the use of the functional conditional average projector has now been updated in this chapter to the RG analysis of passive scalar advection.

6.1 Passive scalar phenomenology

The evolution of a passive scalar field $\phi(\mathbf{x},t)$ advected by a fluid with velocity field $u_\beta(\mathbf{x},t)$, is given by the differential equation

$$\left[\frac{\partial}{\partial t}+\chi\frac{\partial^2}{\partial x_\beta \partial x_\beta}\right]\phi(\mathbf{x},t)=u_\beta(\mathbf{x},t)\frac{\partial}{\partial x_\beta}\phi(\mathbf{x},t), \tag{6.1}$$

where χ is the molecular diffusivity, the passive scalar analog to the viscosity in the NSE; thus the second term in the square brackets represents the diffusion

term. (6.1) is simply the diffusion equation with an extra advective term representing the action of the fluid on the passive scalar field $\phi(\mathbf{x},t)$. The evolution of the velocity field $u_\alpha(\mathbf{x},t)$ is prescribed by the NSE

$$\left[\frac{\partial}{\partial t}+\nu\frac{\partial^2}{\partial x_\beta \partial x_\beta}\right]u_\alpha(\mathbf{x},t)=u_\beta(\mathbf{x},t)\frac{\partial}{\partial x_\beta}u_\alpha(\mathbf{x},t)+f_\alpha(\mathbf{x},t) \tag{6.2}$$

with the incompressibility condition

$$\frac{\partial}{\partial x_\beta}u_\beta(\mathbf{x},t)=0\ , \tag{6.3}$$

and the velocity field statistics are assumed to be homogeneous, isotropic and stationary.

The term on the RHS of equation (6.1) can be considered to be behaving like a non-linear forcing term advecting the scalar field. However, this picture can be slightly misleading, as this 'forcing' term does not do any work on the system; it is simply responsible for the scalar field taking a 'free-ride' on the velocity field. $\phi(\mathbf{k},t)$ represents any scalar field riding on the velocity field i.e. it could be a temperature field, mass or density of particles/contaminant field etc. And it is passive in the sense that it has no effect on the velocity field. such that the scalar field is coupled/slaved to the velocity field, but the velocity field is not coupled to the scalar field. The arbitrariness of the units of the passive scalar field (depending on what the scalar field is) immediately implies that the molecular diffusivity has the same units as the kinematic viscosity in the NSE (6.2).

As before, we will be working in Fourier space. Accordingly, transforming equations (6.1)-(6.3) gives

$$\left[\frac{\partial}{\partial t}+\chi k^2\right]\phi(\mathbf{k},t)=-ik_\beta\int d^3j u_\beta(\mathbf{j},t)\phi(\mathbf{k}-\mathbf{j},t), \tag{6.4}$$

$$\left[\frac{\partial}{\partial t}+\nu k^2\right]u_\alpha(\mathbf{k},t)=M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j u_\beta(\mathbf{j},t)u_\gamma(\mathbf{k}-\mathbf{j},t)+f_\alpha(\mathbf{k},t), \tag{6.5}$$

and

$$k_\beta u_\beta(\mathbf{k},t). \tag{6.6}$$

Looking at the convolution term in equation (6.4) we can see the extent of the nonlinearity as only being dependent upon the nonlinear nature of the velocity

field which is advecting it. Terms like these are known as bilinear.

Before we move on to implementing the RG procedure on the above equations, we will briefly review some useful parameters. Although we will not be needing it, the Reynolds number equivalent for the passive scalar field is the Peclet number [84] and is given by

$$Pe = \frac{Vl}{\chi}, \tag{6.7}$$

where l is some length scale of the passive scalar field fluctuations and V is some associated velocity scale. The scalar variance spectrum, $E_\phi(k,t)$, is given by

$$\langle \phi(\mathbf{k},t)\phi(\mathbf{k}',t)\rangle = \frac{E_\phi(k,t)}{2\pi k^2}\delta(\mathbf{k}+\mathbf{k}'), \tag{6.8}$$

and the scalar dissipation rate is

$$\varepsilon_\phi = \int_0^\infty 2\chi k^2 E_\phi(k,t)dk. \tag{6.9}$$

The equivalent form for the Kolmogorov dissipation wavenumber is the Corrsin convective wavenumber k_ϕ given by

$$\begin{aligned} k_\phi = \left(\frac{\varepsilon}{\chi^3}\right)^{1/4} &= \left(\frac{\nu}{\chi}\right)^{3/4} k_d \\ &= Pr^{3/4} k_d, \end{aligned} \tag{6.10}$$

where ε is the normal turbulent energy dissipation rate and Pr is the Prandtl or Schmidt number [97] which is the ratio of viscous effects to molecular diffusion effects, and from (6.10) is given by

$$Pr = \frac{\nu}{\chi} = \left(\frac{k_\phi}{k_d}\right)^{4/3}. \tag{6.11}$$

Equation (6.11) shows that if viscous effects are stronger than the molecular diffusion effects of the scalar field, then $k_\phi > k_d$ and the normal Kolmogorov wavenumber k_d is the characteristic wavenumber at which the passive scalar 'energy' dissipates. Whilst if viscous effects are weaker then it will be k_ϕ at which dissipation effects kick in for the passive scalar field. Here, we refer to the scalar variance spectrum as the passive scalar 'energy' spectrum because we will be treating it like our earlier velocity spectral kinetic energy term. However, we should remember that it is in no other ways similar to energy in the

conventional sense as can be seen by a dimensional check. The units of the scalar variance spectrum depend upon what the passive scalar field represents i.e. for the temperature field this would have units $metre \times kelvin^2$. We must also remember that the above analysis of at which wavenumber dissipation begins to dominate does not effect the velocity field kinetic energy which always has k_d as its characteristic dissipation wavenumber. The above analysis only applies to the passive scalar field.

Similar to the case for the velocity field where there exists a universal inertial range, there also exists a similar range for the passive scalar problem. This is known as the inertial-convective range and Oboukhov and Corrsin proposed that the scalar variance spectrum $E_\phi(k)$ should be proportional to the energy spectrum in the following way

$$E_\phi(k) \propto \frac{\varepsilon_\phi}{\varepsilon} E(k), \tag{6.12}$$

which when substituted for the form of the Kolmogorov spectrum, leads onto the power law behaviour of the scalar variance spectrum in the inertial-convective range given by

$$E_\phi(k) = \beta \varepsilon_\phi \varepsilon^{-1/3} k^{-5/3}, \tag{6.13}$$

where β is known as the Oboukhov-Corrsin constant.

Finally, for the purposes of the RG calculation we will be defining the UV cut-off wavenumber $k_{\phi 0}$ in a similar way set in Chapter 2, where we require the cut-off to preserve the scalar dissipation rate

$$\varepsilon_\phi = \int_0^\infty 2\chi k^2 E_\phi(k,t) dk \simeq \int_0^{k_{\phi 0}} 2\chi k^2 E_\phi(k,t) dk. \tag{6.14}$$

6.2 Extension of the functional conditional average

Before we begin to reduce the degrees of freedom, we need to expand our definition of the conditional average operation to include the statistics of a passive scalar field so that we have a tool to coarse-grain our system. This task presents some obstacles with respect to ordering of differentiation, permutations etc., which need to be overcome to provide a generalisation of equation

(3.22) for two fields. Equations (3.16) and (3.17) of Chapter 3 show a generalisation of the functional form of the ensemble average of two functionals and provide an initial step of how we may extend this generalisation to the conditional average of two or more different functions. This step concerns formally updating the conditional probability distribution functional in equation (3.22) to include the passive scalar field statistics. After this is done, we extend the notion of the conditional average operation to the case of two fields (one vector field and one scalar field) with the following operator

$$\begin{aligned}
&\widehat{\mathcal{P}}_c^- \left\{ f\left[u_\alpha\left(\mathbf{k},t\right), \phi\left(\mathbf{j},t\right)\right]\right\} \\
&\quad = \frac{1}{N!}\prod_{i=1}^{N}\frac{1}{M!}\prod_{j=1}^{M}\int d^3p_i \int ds_i u_{\sigma_i}^-\left(\mathbf{p}_i,s_i\right)\int d^3q_j\int dr_j \phi^-\left(\mathbf{q}_j,r_j\right)\times \\
&\quad \times \int \mathcal{D}w_\lambda\left(\mathbf{z},\tau\right)\int \mathcal{D}y\left(\mathbf{b},\xi\right) P\left[w_\lambda\left(\mathbf{z},\tau\right), y\left(\mathbf{b},\xi\right)\right]\times \\
&\quad \times \frac{\delta^{N+M} f\left[w_\alpha\left(\mathbf{k},t\right), y\left(\mathbf{j},t\right)\right]}{\delta w_{\sigma_1}^-\left(\mathbf{p}_1,s_1\right)\ldots\delta w_{\sigma_N}^-\left(\mathbf{p}_N,s_N\right)\delta y^-\left(\mathbf{q}_1,r_1\right)\ldots\delta y^-\left(\mathbf{q}_M,r_M\right)},
\end{aligned} \tag{6.15}$$

where it is important to note that the functional differentiation with respect to the scalar field dummy variable y^- is done first, and that this ordering of the differentiation is necessary. This is due to the u^- field being independent of the ϕ^- field. We will now show the application of equation (6.15) with an example of the evaluation of $\hat{\mathcal{P}}_{\mathcal{C}}\left\{u_\alpha^-\left(\mathbf{k},t\right)\phi^-\left(\mathbf{j},t\right)\phi^+\left(\mathbf{l},t\right)\right\}$

$$\begin{aligned}
&\hat{\mathcal{P}}_{\mathcal{C}}\left\{u_\alpha^-\left(\mathbf{k},t\right)\phi^-\left(\mathbf{j},t\right)\phi^+\left(\mathbf{l},t\right)\right\} \\
&\quad = \int d^3p\int ds u_\sigma^-\left(\mathbf{p},s\right)\int d^3q\int dr\phi^-\left(\mathbf{q},r\right)\times \\
&\quad \times \int \mathcal{D}w_\lambda\left(\mathbf{z},\tau\right)\int \mathcal{D}y\left(\mathbf{b},\xi\right) P\left[w_\lambda\left(\mathbf{z},\tau\right), y\left(\mathbf{b},\xi\right)\right]\times \\
&\quad \times \frac{\delta^2\left(w_\alpha^-\left(\mathbf{k},t\right)y^-\left(\mathbf{j},t\right)y^+\left(\mathbf{l},t\right)\right)}{\delta w_\sigma^-\left(\mathbf{p},s\right)\delta y^-\left(\mathbf{q},r\right)},
\end{aligned} \tag{6.16}$$

where we could have expanded ϕ^+ in terms of u^- and ϕ^- to any order using equations (6.4) and (6.5), but have chosen to work at zero order for the purposes of this example. Moving on and evaluating the functional derivative in

equation (6.16) we get

$$\begin{aligned}
&\frac{\delta^2 f\left(w_\alpha^-\left(\mathbf{k},t\right)y^-\left(\mathbf{j},t\right)y^+\left(\mathbf{l},t\right)\right)}{\delta w_\sigma^-\left(\mathbf{p},s\right)\delta y^-\left(\mathbf{q},r\right)} \\
= \;& \frac{\delta}{\delta w_\sigma^-\left(\mathbf{p},s\right)}\left\{\delta(\mathbf{j}-\mathbf{q})\delta(t-r)w_\alpha^-\left(\mathbf{k},t\right)y^+\left(\mathbf{l},t\right)+\right. \\
& +y^-\left(\mathbf{j},t\right)y^+\left(\mathbf{l},t\right)\frac{\delta}{\delta y^-\left(\mathbf{q},r\right)}w_\alpha^-\left(\mathbf{k},t\right)+ \\
& \left.+y^-\left(\mathbf{j},t\right)w_\alpha^-\left(\mathbf{k},t\right)\frac{\delta}{\delta y^-\left(\mathbf{q},r\right)}y^+\left(\mathbf{l},t\right)\right\} \\
= \;& \delta(\mathbf{j}-\mathbf{q})\delta(t-r)\delta_{\alpha\sigma}\delta(\mathbf{k}-\mathbf{p})\delta(t-s)y^+\left(\mathbf{l},t\right), \qquad (6.17)
\end{aligned}$$

where the last two terms on the RHS of the first equality vanish due to a.) w^- being independent of y^- and b.) y^+ being independent of y^- due to zero order approximation of y^+ in term of y^- as mentioned earlier. Substituting the result (6.17) back into equation (6.16) we get the desired result

$$\hat{\mathcal{P}}_{\mathcal{C}}\left\{u_\alpha^-\left(\mathbf{k},t\right)\phi^-\left(\mathbf{j},t\right)\phi^+\left(\mathbf{l},t\right)\right\}=u_\alpha^-\left(\mathbf{k},t\right)\phi^-\left(\mathbf{j},t\right)\left\langle\phi^+\left(\mathbf{l},t\right)\right\rangle. \qquad (6.18)$$

Before we move on we must also mention that the asymptotic freedom operation embedded in the derivation of the conditional average implies that

$$\hat{\mathcal{P}}_{\mathcal{C}}\left\{u^+\phi^+\right\}=\left\langle u^+\right\rangle\left\langle\phi^+\right\rangle, \qquad (6.19)$$

because of the very fact that the u^+ and ϕ^+ fields are assumed to be asymptotically free and hence, statistically independent. We now have the tools to proceed with the RG calculation.

6.3 Conditional mode elimination

As in the previous chapter, in order to simplify the calculation, we will not go to a dimensionless form of the equations but will assume implicitly that we are evaluating the RG calculation to second order in local Reynolds number in both the velocity (NSE) and passive scalar equations; remembering that the local Reynolds number is the expansion parameter and is small in the sub-grid regions of the respective spectra as discussed in Chapter 2.

As before, we begin by partitioning the relevant equations into low and high

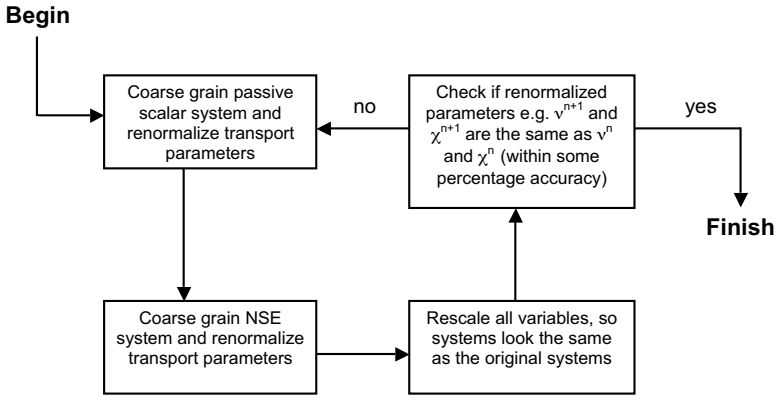

Figure 6.1: RG algorithm for the passive scalar system.

wavenumber parts. We will only focus on equation (6.4) here, as the analysis for the NSE (equation (6.5)) is the same as Chapter 4. What we have to remember is that our RG procedure will be renormalizing both the diffusivity and the viscosity together. To illustrate this procedure, the RG algorithm for the passive scalar advection system is shown in fig. 6.1.

Partitioning equation (6.4) we obtain

$$\begin{aligned}\left[\frac{\partial}{\partial t}+\chi_0(k)k^2\right]\phi^-(\mathbf{k},t) &= -ik_\beta^- \int d^3j \left[u_\beta^-(\mathbf{j},t)\phi^-(\mathbf{k}-\mathbf{j},t)\right. \\ &\quad +u_\beta^-(\mathbf{j},t)\phi^+(\mathbf{k}-\mathbf{j},t)+u_\beta^+(\mathbf{j},t)\phi^-(\mathbf{k}-\mathbf{j},t) \\ &\quad \left.+u_\beta^+(\mathbf{j},t)\phi^+(\mathbf{k}-\mathbf{j},t)\right], \qquad (6.20)\end{aligned}$$

for the low wavenumber part and

$$\begin{aligned}\left[\frac{\partial}{\partial t}+\chi_0(k)k^2\right]\phi^+(\mathbf{k},t) &= -ik_\beta^+ \int d^3j \left[u_\beta^-(\mathbf{j},t)\phi^-(\mathbf{k}-\mathbf{j},t)\right. \\ &\quad +u_\beta^-(\mathbf{j},t)\phi^+(\mathbf{k}-\mathbf{j},t)+u_\beta^+(\mathbf{j},t)\phi^-(\mathbf{k}-\mathbf{j},t) \\ &\quad \left.+u_\beta^+(\mathbf{j},t)\phi^+(\mathbf{k}-\mathbf{j},t)\right], \qquad (6.21)\end{aligned}$$

for the high wavenumber part, and where we have labeled the diffusivity with a subscript zero and made it a function of wavenumber to facilitate the RG calculation as in Chapter 4. Applying the CA on equation (6.20) for the low

wavenumber modes we obtain

$$\left[\frac{\partial}{\partial t}+\chi_0(k)k^2\right]\phi^-(\mathbf{k},t) = -ik^-_\beta\int d^3j\left[u^-_\beta(\mathbf{j},t)\phi^-(\mathbf{k}-\mathbf{j},t)\right.$$
$$+\hat{\mathcal{P}}_{\mathcal{C}}\left\{u^-_\beta(\mathbf{j},t)\phi^+(\mathbf{k}-\mathbf{j},t)\right\}+\hat{\mathcal{P}}_{\mathcal{C}}\left\{u^+_\beta(\mathbf{j},t)\phi^-(\mathbf{k}-\mathbf{j},t)\right\}$$
$$\left.+\hat{\mathcal{P}}_{\mathcal{C}}\left\{u^+_\beta(\mathbf{j},t)\phi^+(\mathbf{k}-\mathbf{j},t)\right\}\right]. \quad (6.22)$$

In accordance with the earlier RG calculation presented in Chapter 4 we will evaluate the CA of the cross-terms, as they are presented above, at first order. As the CA of the cross-terms result in zero at first order, we are left with

$$\left[\frac{\partial}{\partial t}+\chi_0(k)k^2\right]\phi^-(\mathbf{k},t) = -ik^-_\beta\int d^3j\left[u^-_\beta(\mathbf{j},t)\phi^-(\mathbf{k}-\mathbf{j},t)\right.$$
$$\left.+\hat{\mathcal{P}}_{\mathcal{C}}\left\{u^+_\beta(\mathbf{j},t)\phi^+(\mathbf{k}-\mathbf{j},t)\right\}\right]. \quad (6.23)$$

To evaluate the CA of the Reynolds-like term we construct a dynamical equation for $u^-_\beta(\mathbf{j},t)$ using (6.5), and multiply this by $\phi^+(\mathbf{k}-\mathbf{j},t)$. Then we construct an equation for $\phi^+(\mathbf{k}-\mathbf{j},t)$ using (6.4) and multiply this by $u^-_\beta(\mathbf{j},t)$. Adding the resultant equations gives

$$\left[\frac{\partial}{\partial t}+\nu_0(j)j^2+\chi_0\left(|\mathbf{k}-\mathbf{j}|\right)|\mathbf{k}-\mathbf{j}|^2\right]u^+_\beta(\mathbf{j},t)\phi^+(\mathbf{k}-\mathbf{j},t)$$
$$=-i\left(k^+_\rho-j^+_\rho\right)\int d^3p\left[u^-_\rho(\mathbf{p},t)\phi^-(\mathbf{k}-\mathbf{j}-\mathbf{p},t)u^+_\beta(\mathbf{j},t)+\right.$$
$$+u^-_\rho(\mathbf{p},t)\phi^+(\mathbf{k}-\mathbf{j}-\mathbf{p},t)u^+_\beta(\mathbf{j},t)+u^+_\rho(\mathbf{p},t)\phi^-(\mathbf{k}-\mathbf{j}-\mathbf{p},t)u^+_\beta(\mathbf{j},t)+$$
$$\left.+u^+_\rho(\mathbf{p},t)\phi^+(\mathbf{k}-\mathbf{j}-\mathbf{p},t)u^+_\beta(\mathbf{j},t)\right]+$$
$$+M^+_{\beta\sigma\delta}(\mathbf{j})\int d^3p\left[u^-_\sigma(\mathbf{p},t)u^-_\delta(\mathbf{j}-\mathbf{p},t)\phi^+(\mathbf{k}-\mathbf{j},t)+\right.$$
$$+2u^-_\sigma(\mathbf{p},t)u^+_\delta(\mathbf{j}-\mathbf{p},t)\phi^+(\mathbf{k}-\mathbf{j},t)+$$
$$\left.+u^+_\sigma(\mathbf{p},t)u^+_\delta(\mathbf{j}-\mathbf{p},t)\phi^+(\mathbf{k}-\mathbf{j},t)\right]. \quad (6.24)$$

Inverting the linear operator on the LHS of equation (6.24), applying the CA operation and evaluating the resultant terms will leave only one relevant term

in the order of approximation we are working

$$\begin{aligned}\hat{\mathcal{P}}_{\mathcal{C}}\left\{u_\beta^+(\mathbf{j},t)\phi^+(\mathbf{k}-\mathbf{j},t)\right\} &= -i\left(k_\rho^+ - j_\rho^+\right)\int_{-\infty}^{t} ds e^{-(\nu_0(j)j^2+\chi_0(|\mathbf{k}-\mathbf{j}|)|\mathbf{k}-\mathbf{j}|^2)(t-s)} \times \\ &\times \int d^3p \phi^-(\mathbf{k}-\mathbf{j}-\mathbf{p},s)\left\langle u_\rho^+(\mathbf{p},s)u_\beta^+(\mathbf{j},s)\right\rangle \\ &= -i\left(k_\rho^+ - j_\rho^+\right)\int d^3p \frac{\left\langle u_\rho^+(\mathbf{p},t)u_\beta^+(\mathbf{j},t)\right\rangle \phi^-(\mathbf{k}-\mathbf{j}-\mathbf{p},t)}{\nu_0(j)j^2+\chi_0\left(|\mathbf{k}-\mathbf{j}|\right)|\mathbf{k}-\mathbf{j}|^2-\chi_0(k)k^2}, \end{aligned} \tag{6.25}$$

where in the last line we have evaluated the time-history integral as in Chapter 4. Integrating over $\mathbf{j}$ and multiplying by a factor $-ik_\beta^-$ gives the full Reynolds-like term that was required initially

$$\begin{aligned}-ik_\beta^-\int d^3j \hat{\mathcal{P}}_{\mathcal{C}}\left\{u_\beta^+(\mathbf{j},t)\phi^+(\mathbf{k}-\mathbf{j},t)\right\} &= -\int d^3j k_\beta^-\left(k_\rho^+ - j_\rho^+\right)\times \\ &\times \int d^3p \frac{\left\langle u_\rho^+(\mathbf{p},t)u_\beta^+(\mathbf{j},t)\right\rangle \phi^-(\mathbf{k}-\mathbf{j}-\mathbf{p},t)}{\nu_0(j)j^2+\chi_0\left(|\mathbf{k}-\mathbf{j}|\right)|\mathbf{k}-\mathbf{j}|^2-\chi_0(k)k^2} \\ &= -\int d^3j \int d^3p \frac{k_\beta^- j_\rho^+\left\langle u_\rho^+(\mathbf{p},t)u_\beta^+(\mathbf{k}-\mathbf{j},t)\right\rangle \phi^-(\mathbf{j}-\mathbf{p},t)}{\chi_0(j)j^2+\nu_0\left(|\mathbf{k}-\mathbf{j}|\right)|\mathbf{k}-\mathbf{j}|^2-\chi_0(k)k^2}, \end{aligned} \tag{6.26}$$

where in the last line we have used the symmetry property of convolutions [98]

$$\int_{-\infty}^{\infty} dx f(x)g(y-x) = \int_{-\infty}^{\infty} dx f(y-x)g(x). \tag{6.27}$$

Reducing and evaluating the isotropic tensors in equation (6.26) gives us

$$\begin{aligned}&-ik_\beta^-\int d^3j \hat{\mathcal{P}}_{\mathcal{C}}\left\{u_\beta^+(\mathbf{j},t)\phi^+(\mathbf{k}-\mathbf{j},t)\right\} \\ &= -\int d^3j \frac{R(\mathbf{k},\mathbf{j})Q^+(|\mathbf{k}-\mathbf{j}|)}{\chi_0(j)j^2+\nu_0\left(|\mathbf{k}-\mathbf{j}|\right)|\mathbf{k}-\mathbf{j}|^2-\chi_0(k)k^2}\phi^-(\mathbf{k},t), \end{aligned} \tag{6.28}$$

where

$$\begin{aligned} R(\mathbf{k},\mathbf{j}) &= P_{\rho\beta}(\mathbf{k}-\mathbf{j})k_\beta j_\rho \\ &= \frac{k^2j^2-k^2j^2\mu^2}{k^2+j^2-2kj\mu}. \end{aligned} \tag{6.29}$$

When (6.28) is substituted back into equation (6.23) we obtain

$$\left[\frac{\partial}{\partial t}+\chi_1(k)k^2\right]\phi^-(\mathbf{k},t) = -ik_\beta^-\int d^3j u_\beta^-(\mathbf{j},t)\phi^-(\mathbf{k}-\mathbf{j},t), \tag{6.30}$$

defined in the interval $0 \leq k \leq k_{\phi 1}$, and where

$$\chi_1(k) = \chi_0(k) + \delta\chi_0(k), \tag{6.31}$$

and the increment to the diffusivity is given by

$$\delta\chi_0(k) = \frac{1}{k^2}\int d^3j \frac{R(\mathbf{k},\mathbf{j})Q^+(|\mathbf{k}-\mathbf{j}|)}{\chi_0(j)j^2 + \nu_0\left(|\mathbf{k}-\mathbf{j}|\right)|\mathbf{k}-\mathbf{j}|^2 - \chi_0(k)k^2}. \tag{6.32}$$

This, along with the analysis in Chapter 4 for the NSE velocity field, completes the first shell elimination for the passive scalar advection dynamical system.

6.4 The RG equations

The second RG procedure of rescaling is done in a similar way to the analysis in Chapter 4, and thus will not be repeated here. However, one important thing to note is that the diffusivity will scale in a similar way to the viscosity

$$\tilde{\chi}_n(k') = \alpha^{-1/2}\varepsilon^{1/3}k_n^{-4/3}\chi(k). \tag{6.33}$$

Similar to the scaled viscosity, relation (6.33) ensures self-consistency in that the increment to the diffusivity scales in the same way as the diffusivity, and that the scalar flux ε_ϕ is left invariant to the scaling.

We will complete the RG analysis for the passive scalar advection system by stating the final results for the RG recursion relations after making the equations dimensionless and generalising to the n^{th} shell removal

$$\tilde{\nu}_{n+1}(k') = h^{4/3}\tilde{\nu}_n(hk') + h^{-4/3}\delta\tilde{\nu}_n(k'), \tag{6.34}$$

$$\tilde{\chi}_{n+1}(k') = h^{4/3}\tilde{\chi}_n(hk') + h^{-4/3}\delta\tilde{\chi}_n(k'), \tag{6.35}$$

where

$$\delta\nu_n(k') = \frac{1}{4\pi k'^2}\int d^3j' \frac{L(\mathbf{k}',\mathbf{j}')\left[h^{11/3} - \left(|\mathbf{k}'-\mathbf{j}'| - h^{-1}\right)\frac{11}{3}h^{14/3}\right]}{\tilde{\nu}_n(hj')j'^2 + \tilde{\nu}_n\left(h\,|\mathbf{k}'-\mathbf{j}'|\right)|\mathbf{k}'-\mathbf{j}'|^2 - \tilde{\nu}_n(hk')k'^2}, \tag{6.36}$$

$$\delta\chi(k|k_n) = \frac{1}{4\pi k'^2}\int d^3j' \frac{R(\mathbf{k}',\mathbf{j}')\left[h^{11/3} - (|\mathbf{k}'-\mathbf{j}'| - h^{-1})\,\frac{11}{3}h^{14/3}\right]}{\tilde{\chi}_n(hj')j'^2 + \tilde{\nu}_n(h\,|\mathbf{k}'-\mathbf{j}'|)\,|\mathbf{k}'-\mathbf{j}'|^2 - \tilde{\chi}(hk')k'^2}. \quad (6.37)$$

Note that there was essentially no other extra work involved other than the analysis in Chapter 4, to obtain the above scaled relations.

6.5 Results

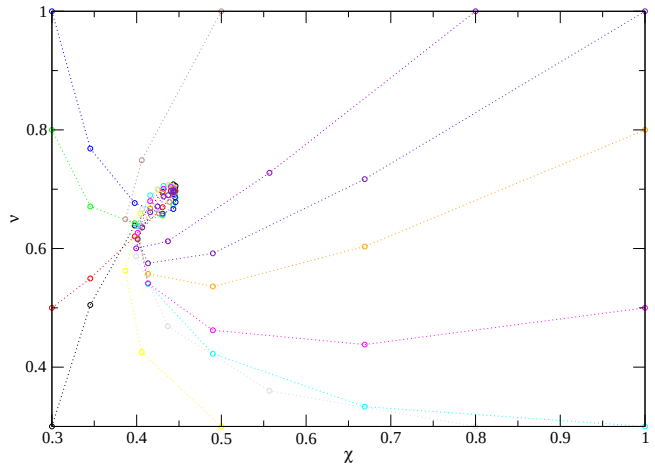

Figure 6.2: RG map showing the iteration and convergence of the scaled renormalized viscosity and diffusivity to the fixed point for $k' = 0.1$ and $\eta = 0.3$; the convergence indicates the occurrence of universal behaviour in not only these parameters, but also in the renormalized Prandtl number.

Iterating the RG equations (6.34)-(6.37), we find that our calculation converges upon a fixed point in the renormalized diffusivity and viscosity. This can be seen in the RG map shown in figure 6.2. The calculation converges upon the same fixed point for several initial values of the scaled viscosity and diffusivity indicating a stable fixed point. This not only points to universal behaviour in these two renormalized parameters, but also indicates that the renormalized Prandtl number reaches a fixed point and this exhibits universality also.

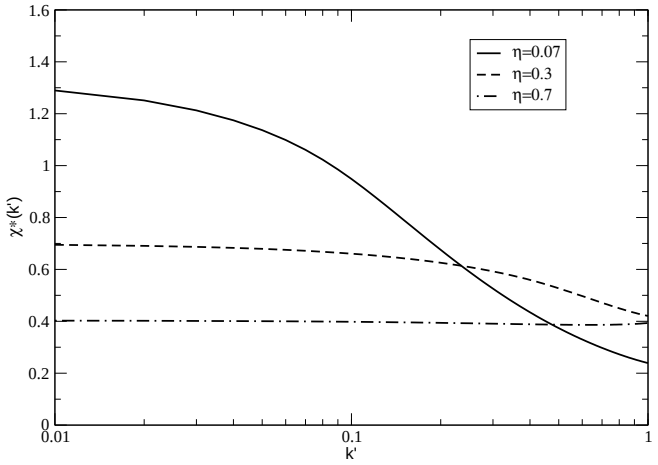

Figure 6.3: Scaled fixed point renormalized diffusivity for three values of the shell width parameter η.

6.5.1 Scale-invariant diffusivity

Figure 6.3 shows the form of the fixed point renormalized diffusivity for three values of the shell width parameter η. The fixed point renormalized viscosity has exactly the same results as in Chapter 4 due to the scalar field being passive and not affecting the velocity field. The fixed point diffusivity exhibits the same sort of features as the renormalized fixed point viscosity. In particular we see that the diffusivity tends to become independent of the wavenumber for large values of η. This is not too surprising as this is a result of scale separation between the high and low wavenumber mode statistics which in turn is due to the asymptotic freedom assumption embedded in the conditional average operation. This scale separation is also seen in the behaviour of the fixed point renormalized diffusivity reaching a constant value in the limit $k' \to 0$; as in the case for the fixed point renormalized viscosity.

6.5.2 Scale-invariant Prandtl number

To investigate further the scale free behaviour of the Prandtl number Pr (see relation (6.11)) we have constructed a wavenumber dependent Pr from the

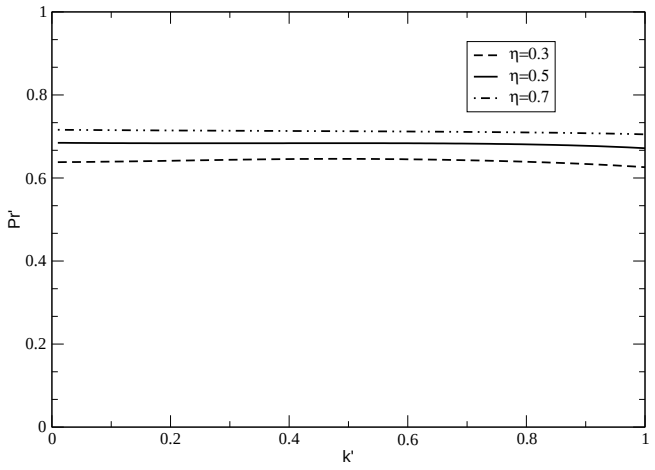

Figure 6.4: Fixed point renormalized Prandtl number Pr for three values of the shell width parameter η. We have used the scaled values of the viscosity and diffusivity; however, since they scale in the same way, it does not matter if we construct Pr from scaled or unscaled values.

fixed point renormalized viscosities and diffusivities. The results for the fixed point renormalized Pr are shown in figure 6.4 for three different values of the shell width η. It is quite clear to see that the fixed point Pr is pretty much constant over the whole k' range, and for all three values of η lie around the value of $Pr = 0.7 \pm 0.05$. A comprehensive survey of both experimental and computational (DNS) results was done by Lin, Chang and Wang [99]. This study showed that there is a trend of the *turbulent* Prandtl number Pr^T (equivalent to our renormalized one) to tend to unity as the turbulent Peclet number tends to infinity, irrespective of what the raw (unrenormalized) Prandtl number given by relation (6.11) is. This picture of parameters exhibiting scale-free universal behaviour as the Reynolds/Peclet numbers become very large is what our study is trying to ascertain. Although the study of Lin *et. al.* indicates the tendency for Pr^T to reach unity, the value actually measured in experiments by Fulachier and Dumas [100] is in the range $0.6 \leq Pr^T \leq 0.8$, which agrees very well with the values obtained from our RG calculation. Lin *et. al.* suggest that the reason why Pr^T always tends to unity as the turbulent Peclet number becomes very large, is that at that time the scalar field turbulence is completely convection dominated and the scalar field responds immediately to any changes in the ve-

locity field. Similar arguments have also been made by Hinze [101], Tennekes & Lumley [97], and the study of Zhou *et. al.* [102] on their RG scheme and its applications to passive scalar advection. The work of the latter is of particular interest, as they also use a recursive RG scheme. Zhou *et. al* also have a value of $Pr^T \sim 0.7$ in the constant part of their plots of Pr^T. They also obtain a mild cusp; however this could simply be an artificial result of the taming of their divergences, the form of which has already been discussed in the previous chapter.

6.5.3 Comparison with the work of Rose

In one of the earliest attempts to apply RG to Navier-Stokes like systems, Rose [58] studied the case of the advection of a passive scalar by a randomly prescribed frozen velocity field. We will make some attempts at comparing the results presented in this chapter with the work of Rose and others. Before we move on to do this some points need to be made on the major differences between these two pieces of work.

Firstly, Rose's velocity field is prescribed by Gaussian statistics and is considered frozen, so that it does not vary in time. So unlike the work presented here, Rose has no equation of motion for the velocity field. As a result of this, Rose retains only the effects of the cross-term in his calculations and not the Reynolds term, in the passive scalar equation. Secondly, Rose retains the triple nonlinearity in the equation of motion which contributes to the renormalized diffusivity at every step of the RG calculation. Lastly, due to the artificiality of studying a passive scalar advected by a frozen Gaussian velocity field, Rose's results do not apply to real turbulent flows. As a result Rose has no calculation for the Oboukhov-Corrsin constant β. What we may compare, however, is the form of Rose's fixed point renormalized diffusivity. Rose denotes this quantity by β (not to be confused with our Oboukhov-Corrsin constant) and denotes the shell width parameter as f. We used Rose's results for $f = 0.3$ presented in table 2 in [58], and compared them with our results for $\eta = 0.3$. These results can be seen in figure 6.5.

Figure 6.5 shows that Rose's results for the wavenumber dependent fixed point renormalized diffusivity are qualitatively very similar to the work presented in this Chapter. At first this might seem surprising with Rose's assumptions on

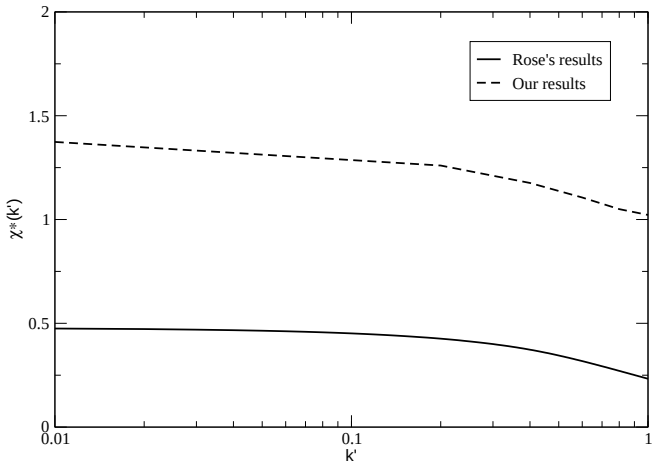

Figure 6.5: Plot comparing our fixed point renormalized diffusivity with Rose's results for a shell width parameter of $\eta = 0.3$.

the statistics of the velocity field, but a further study will show that this is not necessarily so. The velocity field essentially only comes into the calculation when providing a form for the energy spectrum and the form for the viscosity at each stage of the RG calculation. Since in both studies the energy spectrum is assumed to be of the Kolmogorov power law form this should not effect the comparison. The viscosity in our calculation is reasonably constant for most values of η so this should essentially not effect the qualitative nature of the renormalized diffusivity. A possible criticism of this comparison could be that the comparison is not valid as we are including the effects of the Reynolds-term, whilst Rose includes the effects of the cross-term only. However as can be seen by the eddy-viscosity and diffusivity of Zhou, Vahala and Hossain [102, 65] who basically do a similar calculation to Rose but generalised to actual turbulence, the qualitative form of their **renormalized** transport quantities are essentially similar to our work. We emphasize the word 'renormalized' here because as discussed in chapters 2 and 5, they also add on the effect of the cumulative triple non-linearity using a quasi-normal hypothesis, to get an effective viscosity and diffusivity.

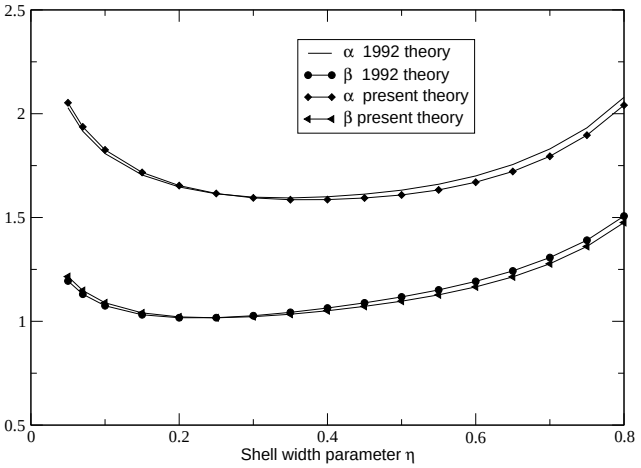

Figure 6.6: Variation of the Kolmogorov and Oboukhov-Corrsin constants with shell width η; shown for the older 1992 two-field theory of McComb & Watt and the new current theory with no Markovianization.

6.5.4 Prediction of Oboukhov-Corrsin constant

Lastly, we may also calculate from our results, a value for the Oboukhov-Corrsin constant, β, in a similar way to how the Kolmogorov constant was calculated. This is done by first substituting equation (6.13) for the Oboukhov-Corrsin scalar variance spectrum into the renormalized version of relation (6.14) after the RG procedure has finished, to obtain

$$\varepsilon_\phi \simeq \int_0^{k_\phi^*} 2\chi^*(k)\beta\varepsilon_\phi\varepsilon^{-1/3}k^{1/3}dk, \tag{6.38}$$

where k_ϕ^* and $\chi^*(k)$ corresponds to the cut-off wavenumber and enhanced diffusivity respectively, at the step where we stop the RG calculation. We then make the above relation dimensionless by scaling on the wavenumber scale k_ϕ^* and then rearranging to obtain an expression for β

$$\beta \simeq \left(\int_0^1 2\alpha^{1/2}\tilde{\chi}^*(k')k'^{1/3}dk'\right)^{-1}. \tag{6.39}$$

Notice that this expression involves the Kolmogorov constant, α, due to the scaling for χ. Using our results for the renormalized diffusivity $\tilde{\chi}^*(k')$ and the corresponding value of α for a certain value of the shell width η, we can calculate β. The results for this calculation are shown in figure 6.6. The results for α are also shown repeated along side these results. We see a plateau of validity for our calculation for $0.15 \leq \eta \leq 0.30$ where the value of β does not change appreciably. In this interval the Oboukhov-Corrsin constant takes the value of $\beta = 1.03 \pm 0.02$. The recent high-end DNS, with up to 1024^3 grid points, done by Watanabe and Gotoh found the value of $\beta = 0.68 \pm 0.04$ in their comparatively large inertial-convective range. This confirms the previous experimental survey of Hill (1978) who concluded that the $k^{-5/3}$ behaviour is well-supported and that $0.68 < \beta < 0.83$ [5].

Finally as a point of interest, we conclude by mentioning that our results are in very good agreement with the results of the Yakhot & Orszag RG predictions for the Kolmogorov α, Oboukhov-Corrsin β and turbulent Prandtl Pr^T numbers in the inertial-convective range. Although this is not a robust benchmark, it is worth remembering that the initial success and burst of activity in the Yakhot & Orszag school of 'RG' was riding on the excellent agreement that their theory had with experimental values. In particular they derived a relation between all these parameters [103] given by $\beta = \alpha Pr^T$. The realization that our RG scheme does not possess many of the problems of the theory of Yakhot & Orszag and yet obtains the very good values of the latter theory indicates some promise.

Part II

Renormalized Perturbation Theories

Chapter 7

Renormalized perturbation theories and Eulerian closures

The problem of closing the NSE moment hierarchy was touched upon in the Introduction and it was suggested that renormalized perturbation theories (RPT's) could help in achieving this. In this chapter, we begin by a light introduction to one of the attempts at closing the NSE moment hierarchy pre-RPT: the quasi-normal (QN) approximation. The purpose of this introduction will be to formally set the scene to introducing the perturbation series of the NSE around a Gaussian solution. From this we move onto looking at the techniques of how this series can be partially summed via the introduction of renormalized quantities, and so see why the QN approximation failed. This will be done by loosely following the Wyld formalism for RPT's. The formal renormalization procedure will be followed by reviewing some of the main Eulerian RPT closures; concentrating on the two-time closures. Finally, we finish by reviewing the efficacy and practicalities of computing these closures and reviewing the single-time Markovian closures constructed as a result of this.

7.1 The quasi-normal approximation

In the introduction it was mentioned that the problem of closing the moment hierarchy amounted to a way of approximating a relationship between moments of different order. One of the simplest ways of doing this was proposed by Proudman and Reid [15] and Tatsumi [16]. They assumed that all

fourth order moments may be treated as being Gaussian. Several years later this quasi-normal assumption was given support from experimental measurements [5, 104, 105]. The assumption of quasi-normality immediately closes the moment hierarchy due to the property of even Gaussian moments being factorisable; fourth order moments can be reduced to second order as

$$\begin{aligned}\langle ABCD\rangle &= \langle AB\rangle\langle CD\rangle + \langle AC\rangle\langle BD\rangle + \\ &\quad + \langle AD\rangle\langle BC\rangle . \end{aligned} \tag{7.1}$$

This result is a particular case of what is sometimes known as Wick's theorem. The QN theory obviously does not extend to moments of any order, as Gaussianity would also mean all of the moments are identically zero due to all odd order Gaussian moments being zero and the NSE moment hierarchy connecting moments of all orders.

We begin the QN approximation by writing the second order moment equation

$$\begin{aligned}\left[\frac{\partial}{\partial t} + 2\nu k^2\right] Q(k,t) &= \frac{1}{2} M_{\alpha\beta\gamma}(\mathbf{k}) \left\{ \int d^3 j Q_{\alpha\beta\gamma}(-\mathbf{k}, \mathbf{j}, \mathbf{k}-\mathbf{j}; t) \right. \\ &\quad \left. - \int d^3 j Q_{\alpha\beta\gamma}(\mathbf{k}, -\mathbf{j}, -\mathbf{k}+\mathbf{j}; t) \right\}, \end{aligned} \tag{7.2}$$

which is obtained by substituting relation (1.20) into equation (1.29) and taking the trace, and where

$$Q_{\alpha\beta\gamma}(-\mathbf{k}, \mathbf{j}, \mathbf{k}-\mathbf{j}; t) = \langle u_\alpha(-\mathbf{k}, t)\, u_\beta(\mathbf{j}, t) u_\gamma(\mathbf{k}-\mathbf{j}, t)\rangle . \tag{7.3}$$

The next step is to substitute for the third order moments. This is done by constructing an equation in a similar method to the way equation (1.29) was obtained

$$\begin{aligned}&\left[\frac{\partial}{\partial t} + \nu\left(k^2 + j^2 + |\mathbf{k}-\mathbf{j}|^2\right)\right] Q_{\alpha\beta\gamma}(-\mathbf{k}, \mathbf{j}, \mathbf{k}-\mathbf{j}; t) \\ &= M_{\alpha\sigma\delta}(-\mathbf{k}) \int d^3 p Q_{\beta\gamma\sigma\delta}(\mathbf{j}, \mathbf{k}-\mathbf{j}, -\mathbf{p}, -\mathbf{k}+\mathbf{p}; t) \\ &+ M_{\beta\sigma\delta}(\mathbf{j}) \int d^3 p Q_{\alpha\gamma\sigma\delta}(-\mathbf{k}, \mathbf{k}-\mathbf{j}, \mathbf{p}, \mathbf{j}-\mathbf{p}; t) \\ &+ M_{\gamma\sigma\delta}(\mathbf{k}-\mathbf{j}) \int d^3 p Q_{\alpha\beta\sigma\delta}(-\mathbf{k}, \mathbf{j}, \mathbf{p}, \mathbf{k}-\mathbf{j}-\mathbf{p}; t), \end{aligned} \tag{7.4}$$

where

$$Q_{\beta\gamma\sigma\delta}(\mathbf{j},\mathbf{k}-\mathbf{j},-\mathbf{p},-\mathbf{k}+\mathbf{p};t)=\langle u_\beta\left(\mathbf{j},t\right)u_\gamma\left(\mathbf{k}-\mathbf{j},t\right)u_\sigma(-\mathbf{p},t)u_\delta(-\mathbf{k}+\mathbf{p},t)\rangle\,, \tag{7.5}$$

and where only the equation for the first third-order moment on the RHS of (7.2) is given as an example. Using an integrating factor on (7.4) to invert the linear operator on the LHS, gives

$$\begin{aligned} Q_{\alpha\beta\gamma}(-\mathbf{k},\mathbf{j},\mathbf{k}-\mathbf{j};t) &= \int_{-\infty}^{t} dt' \exp\left\{-\nu\left(k^2+j^2+|\mathbf{k}-\mathbf{j}|^2\right)[t-t']\right\}\times \\ &\quad \times RHS(7.4)\{t\rightarrow t'\}, \end{aligned} \tag{7.6}$$

where the notation $\{t\rightarrow t'\}$ means to replace the argument t on the RHS of (7.4) by t'. This is then substituted in (7.2) and the fourth order moments are factored according to (7.1). We are thus left with an equation which involves second order moments only and have thus closed the moment hierarchy. One is now left with evaluating each of the terms in the resulting expression. This is done by using relation (1.20), which is repeated here for the single-time case

$$\langle u_\alpha(\mathbf{k},t)u_\beta(\mathbf{k}',t)\rangle=\delta\left(\mathbf{k}+\mathbf{k}'\right)P_{\alpha\beta}(\mathbf{k})Q(k;t), \tag{7.7}$$

to reduce the correlation tensors to correlation (scalar) functions. For some second order moments this relation will result in zero e.g. in the factored fourth order moment

$$\begin{aligned} &Q_{\beta\gamma\sigma\delta}(\mathbf{j},\mathbf{k}-\mathbf{j},-\mathbf{p},-\mathbf{k}+\mathbf{p};t) \\ &\quad = Q_{\beta\gamma}(\mathbf{j},\mathbf{k}-\mathbf{j};t)Q_{\sigma\delta}(-\mathbf{p},-\mathbf{k}+\mathbf{p};t) \\ &\quad +Q_{\beta\sigma}(\mathbf{j},-\mathbf{p};t)Q_{\gamma\delta}(\mathbf{k}-\mathbf{j},-\mathbf{k}+\mathbf{p};t) \\ &\quad +Q_{\beta\delta}(\mathbf{j},-\mathbf{k}+\mathbf{p};t)Q_{\gamma\sigma}(\mathbf{k}-\mathbf{j},-\mathbf{p};t)\,, \end{aligned} \tag{7.8}$$

the first term on the RHS gives zero because

$$Q_{\beta\gamma}(\mathbf{j},\mathbf{k}-\mathbf{j};t)=\delta\left(\mathbf{j}+\mathbf{k}-\mathbf{j}\right)P_{\beta\gamma}(\mathbf{j})Q(j;t) \tag{7.9}$$

implies $\mathbf{k}=0$ due to the delta function; and if we remember that this fourth order moment is prefactored by a $M_{\alpha\sigma\delta}(-\mathbf{k})$, this implies $M_{\alpha\sigma\delta}(0)=0$. When all the terms are suitably reduced we find that the equation for the correlation

function (7.2) is reduced to

$$\left(\frac{\partial}{\partial t}+2\nu k^2\right)Q\left(k;t\right)$$
$$=2\int d^3jL\left(\mathbf{k},\mathbf{j}\right)\int_0^t ds\exp\left\{-\nu\left(k^2+j^2+\left|\mathbf{k}-\mathbf{j}\right|^2\right)\left(t-s\right)\right\}\times$$
$$\times Q\left(\left|\mathbf{k}-\mathbf{j}\right|;s\right)\left[Q\left(j;s\right)-Q\left(k;s\right)\right], \tag{7.10}$$

where $L(\mathbf{k},\mathbf{j})$ is given by

$$\begin{aligned} L(\mathbf{k},\mathbf{j}) &= -2M_{\rho\beta\gamma}(\mathbf{k})M_{\beta\rho\delta}(\mathbf{j})P_{\delta\gamma}(\mathbf{k}-\mathbf{j}) \\ &= \frac{-[\mu(k^2+j^2)-kj(1+2\mu^2)](1-\mu^2)kj}{k^2+j^2-2kj\mu}, \end{aligned} \tag{7.11}$$

and where μ is the cosine of the angle between the vectors $\mathbf{k}$ and $\mathbf{j}$. Equation (7.10) is the equation for the spectral energy density or correlation function obtained by the QN approximation. For a prescribed $Q(k;t=0)$, it can be integrated forward in time to yield the free decay of the energy spectrum. However, when this was done numerically, it was found that the QN theory was physically unrealisable due to it yielding an energy spectrum that was in some places negative. Although it is not such a big surprise that an approximation could produce physically unrealisable results, we will come back to providing a reason for this failure and solutions for its correction, in the last section of this chapter.

7.2 The perturbation expansion

Although we will not be explicitly illustrating the perturbation expansion with diagrams here, the basis for the perturbation theory below is the Wyld diagrammatic expansion. For convenience we state again the solenoidal incompressible NSE

$$\left[\frac{\partial}{\partial t}+\nu k^2\right]u_\alpha(\mathbf{k},t)=f_\alpha(\mathbf{k},t)+\lambda M_{\alpha\beta\gamma}(\mathbf{k})\int d^3ju_\beta(\mathbf{j},t)u_\gamma(\mathbf{k}-\mathbf{j},t), \tag{7.12}$$

where λ is a book-keeping parameter used to illustrate the source of turbulence as the non-linearity, and writing the NSE in solenoidal form required the force

to satisfy

$$k_\alpha f_\alpha(\mathbf{k},t) = 0. \tag{7.13}$$

The principal basis and starting point of all perturbation theories is that the exact solution of the system being studied can be expanded in a power series with respect to some parameter of the system, about a certain known solution. The system in our case is the NSE, the required exact solution is $u_\alpha(\mathbf{k},t)$, the known solution being expanded around is $u_\alpha^{(0)}(\mathbf{k},t)$, say, and the relevant parameter characterising the perturbation away from the known solution is λ. Thus the power or perturbation series can be formally written in the following way

$$u_\alpha(\mathbf{k},t) = u_\alpha^{(0)}(\mathbf{k},t) + \lambda u_\alpha^{(1)}(\mathbf{k},t) + \lambda^2 u_\alpha^{(2)}(\mathbf{k},t) + \ldots, \tag{7.14}$$

where $u_\alpha^{(n)}(\mathbf{k},t)$ for all $n \geq 1$ are the amplitudes or coefficients of the expansion which have to be determined.

The known solution that we will be expanding around as suggested by the perturbation series (7.14), is when $\lambda = 0$ i.e. when the non-linearity and hence the turbulence is switched off. This is known as the zero-order solution and represents the purely viscous response to an arbitrary force with prescribed statistics

$$\left[\frac{\partial}{\partial t} + \nu k^2\right] u_\alpha^{(0)}(\mathbf{k},t) = f_\alpha(\mathbf{k},t), \tag{7.15}$$

which is solved by the use of an integrating factor to get

$$\begin{aligned} u_\alpha^{(0)}(\mathbf{k},t) &= \int dt' \exp\left\{-\nu k^2 (t-t')\right\} f_\alpha(\mathbf{k},t') \\ &= \int dt' G_{\alpha\beta}^{(0)}(\mathbf{k};t,t') f_\beta(\mathbf{k},t'), \end{aligned} \tag{7.16}$$

where the response or Green's function[1] is

$$G_{\alpha\beta}^{(0)}(\mathbf{k};t,t') = P_{\alpha\beta}(\mathbf{k}) G^{(0)}(k;t,t'), \tag{7.17}$$

[1]This quantity will be known as the 'Greens', 'response' or 'propagator' function depending upon what method is being reviewed; the important thing to remember is that they all refer to the same quantity.

and

$$G^{(0)}(k;t,t') = \begin{cases} \exp\{-\nu k^2(t-t')\} & t \geq t' \\ 0 & t < t'. \end{cases} \tag{7.18}$$

It is now clear that to prescribe the statistics for the force term will also prescribe the statistics for the zero order velocity field. As mentioned earlier, we know *a posteriori* that the probability distribution function for turbulence velocity spectra, for a single variable at a single point is near-Gaussian; however, general joint probability distributions are found to be of non-Gaussian nature [5, 104, 105]. Even so, in theoretical physics one finds that generally the only functionals amenable to a fair amount of manipulation are of Gaussian form. Therefore in light of this and a loose appeal to the central limit theorem [5], it seems not only pragmatic but quite reasonable that the known solution we should be expanding about is a variable with Gaussian statistics. Accordingly, we shall now prescribe the force to be a random force with Gaussian statistics and an auto-correlation given by

$$\langle f_\alpha(\mathbf{k},t) f_\beta(-\mathbf{k},t')\rangle = P_{\alpha\beta}(\mathbf{k}) w(k,t-t'); \tag{7.19}$$

where we are also taking the force to be isotropic and stationary, and $w(k,t-t')$ is the spectral density function for the random stirring forces.

We begin formally to solve for the $u_\alpha^{(n)}(\mathbf{k},t)$ by substituting the perturbation series (7.14) for $u_\alpha(\mathbf{k},t)$ on both sides of the NSE (7.12), and then equating terms of the same order in λ to produce the hierarchy of equations

$$\left[\frac{\partial}{\partial t} + \nu k^2\right] u_\alpha^{(0)}(\mathbf{k},t) = f_\alpha(\mathbf{k},t), \tag{7.20}$$

$$\left[\frac{\partial}{\partial t} + \nu k^2\right] u_\alpha^{(1)}(\mathbf{k},t) = M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j u_\beta^{(0)}(\mathbf{j},t) u_\gamma^{(0)}(\mathbf{k}-\mathbf{j},t), \tag{7.21}$$

$$\left[\frac{\partial}{\partial t} + \nu k^2\right] u_\alpha^{(2)}(\mathbf{k},t) = 2M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j u_\beta^{(0)}(\mathbf{j},t) u_\gamma^{(1)}(\mathbf{k}-\mathbf{j},t), \tag{7.22}$$

$$\begin{aligned}\left[\frac{\partial}{\partial t} + \nu k^2\right] u_\alpha^{(3)}(\mathbf{k},t) &= 2M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j u_\beta^{(2)}(\mathbf{j},t) u_\gamma^{(0)}(\mathbf{k}-\mathbf{j},t) \\ &\quad + M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j u_\beta^{(1)}(\mathbf{j},t) u_\gamma^{(1)}(\mathbf{k}-\mathbf{j},t),\end{aligned} \tag{7.23}$$

and so on; and where in (7.22) and (7.23) we have the used the property of

convolutions

$$\int d^3j u_\beta(\mathbf{j},t)u_\gamma(\mathbf{k}-\mathbf{j},t) = \int d^3j u_\beta(\mathbf{k}-\mathbf{j},t)u_\gamma(\mathbf{j},t), \tag{7.24}$$

and the indices symmetry

$$M_{\alpha\beta\gamma}(\mathbf{k}) = M_{\alpha\gamma\beta}(\mathbf{k}). \tag{7.25}$$

Inverting the linear operator on the LHS of (7.21) in a similar way to (7.16), one can substitute the resultant expression for $u_\alpha^{(1)}(\mathbf{k},t)$ into (7.22) so that we have $u_\alpha^{(2)}(\mathbf{k},t)$ in terms of the zero-order field $u_\alpha^{(0)}(\mathbf{k},t)$

$$\begin{aligned}\left[\frac{\partial}{\partial t}+\nu k^2\right] u_\alpha^{(2)}(\mathbf{k},t) &= 2M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j u_\beta^{(0)}(\mathbf{j},t)\int dt' G_{\gamma\rho}^{(0)}(\mathbf{k};t,t')\times \\ &\times M_{\rho\sigma\delta}(\mathbf{k}-\mathbf{j})\int d^3p u_\sigma^{(0)}(\mathbf{p},t')u_\delta^{(0)}(\mathbf{k}-\mathbf{j}-\mathbf{p},t'),\end{aligned} \tag{7.26}$$

which for later convenience we will write in the shorthand form

$$L_0 u^{(2)} = 2Mu^{(0)}G^{(0)}Mu^{(0)}u^{(0)} \tag{7.27}$$

Using a similar procedure, we can write $u_\alpha^{(3)}(\mathbf{k},t)$ in terms of the zero-order field to obtain

$$\begin{aligned}L_0 u^{(3)} &= 4MG^{(0)}Mu^{(0)}G^{(0)}Mu^{(0)}u^{(0)}u^{(0)} \\ &\quad +MG^{(0)}Mu^{(0)}u^{(0)}G^{(0)}Mu^{(0)}u^{(0)},\end{aligned} \tag{7.28}$$

and so on for all the terms in the perturbation series.

To look at the statistical properties we use (7.14) to construct the perturbation series for the correlation function $Q(k)$

$$\begin{aligned}Q(k) &= \langle u(k)u(-k)\rangle \\ &= \left\langle u^{(0)}u^{(0)}\right\rangle + 2\lambda\left\langle u^{(0)}u^{(1)}\right\rangle \\ &\quad +\lambda^2\left\langle u^{(1)}u^{(1)}\right\rangle + 2\lambda^2\left\langle u^{(2)}u^{(0)}\right\rangle + \mathcal{O}(\lambda^3)\end{aligned} \tag{7.29}$$

where again, we have used a shorthand notation. The next step is to substitute

for the non-zero-order terms in (7.29) using (7.21) and (7.27), to obtain

$$\begin{aligned} Q(k) &= \left\langle u^{(0)}u^{(0)}\right\rangle + 2\lambda G^{(0)}M\left\langle u^{(0)}u^{(0)}u^{(0)}\right\rangle \\ &\quad +\lambda^2 G^{(0)}MG^{(0)}M\left\langle u^{(0)}u^{(0)}u^{(0)}u^{(0)}\right\rangle \\ &\quad +4\lambda^2 G^{(0)}MG^{(0)}M\left\langle u^{(0)}u^{(0)}u^{(0)}u^{(0)}\right\rangle + \mathcal{O}(\lambda^3), \end{aligned} \tag{7.30}$$

where neither $G^{(0)}$ nor M are affected by the averaging so can be taken out of the angle brackets. To evaluate the moments, we note that the Gaussian property of the zero-order fields mean that all odd order moments vanish, so that

$$\left\langle u^{(0)}u^{(0)}u^{(0)}\right\rangle = 0, \tag{7.31}$$

and all even order moments factorise like (7.1), such that (7.30) reduces to an expression entirely in terms of the zero-order correlation functions $Q^{(0)}$

$$Q(k) = Q^{(0)} + \lambda^2 \sum G^{(0)}MG^{(0)}MQ^{(0)}Q^{(0)} + \mathcal{O}(\lambda^4), \tag{7.32}$$

where the $\sum$ in the second term represents the various permutations in wavenumbers, time arguments and indices.

If λ was a small parameter, then (7.32) would represent a conventional perturbation theory for $Q(k)$ and we could truncate the series at some order with an associated error. However, this is not the case here and as mentioned earlier, λ is just a book-keeping parameter introduced to distinguish different orders in the perturbation series, and is ultimately set to unity at the end of the calculation.

7.3 Partial summation: the renormalization procedure

As it stands (7.32), which we shall call the *primitive* perturbation series, is wildly divergent. This can be seen if we had made the NSE dimensionless as in Chapter 4; the primitive perturbation series would then have been an expansion in Reynolds number, which is always large in turbulent flows. The pioneering approach in handling such a series was laid down by Kraichnan [18] in the form of the direct interaction approximation or DIA. This will be described

in the next section. From a pedagogic view of the topic of RPT, however, it is better to approach the subject matter via the diagrammatic analysis of the NSE first laid down by Wyld [19]. With this in mind, one must confess that this section will not involve any diagrams and thus will also be a poor substitute for an introduction to the renormalization procedure, let alone a detailed account. We aim simply to provide a loose and heuristic reasoning for the renormalization procedure, which, to the unfamiliar reader, can seem to be a gross wave of a magic 'renormalizing' wand. For an excellent and detailed introduction to the Wyld diagram formulation as well as other RPT's one is recommended to see McComb [5] and the original paper by Wyld [19] as well as the later paper of Lee [106].

The use of the methods introduced by Wyld to the field of turbulence study, involves methods similar to those of quantum field theory where terms in a infinite perturbation series are partitioned into terms which can be summed to all orders and other terms which can be summed to provide another infinite series. In fact, much of the terminology used by Wyld is that of field-theory. To explain and illustrate this further, let us start by first looking at the second-order term in the primitive perturbation series given by (7.32) . Now if one looked at the detailed wavenumbers and other arguments, one will see that there exist three permutations of this term. We will write this term with respect to these permutations as

$$\sum G^{(0)}MG^{(0)}MQ^{(0)}Q^{(0)} = [A]_2 + [B]_2 + [C]_2, \tag{7.33}$$

where the subscript 2 refers to the terms being of second-order in the primitive perturbation series. Furthermore we will now be a bit more harsh on the cavalier way in which we have written these terms and demand that it now becomes important in what order we write the constituents of these terms. So, to illustrate we will write the three different permutations *as* permutations in which the order of the quantities matter

$$\begin{aligned} [A]_2 &= G^{(0)} \quad MQ^{(0)}Q^{(0)}M \quad G^{(0)}, \\ [B]_2 &= G^{(0)} \quad MG^{(0)}Q^{(0)}M \quad Q^{(0)}, \\ [C]_2 &= Q^{(0)} \quad MG^{(0)}Q^{(0)}M \quad G^{(0)}, \end{aligned} \tag{7.34}$$

where the reasons for writing the outer quantities slightly separated will be

shown soon. Using (7.16) we can construct an equation for $Q^{(0)}$ in terms of the force auto-correlation, and we can substitute for the zero-order term in (7.32) to write

$$Q^{(0)} = G^{(0)} w G^{(0)} \ , \tag{7.35}$$

where w is from the statistics of the forcing in relation (7.19). Now substituting (7.35) in (7.34) for all cases where $Q^{(0)}$ occurs as an outer quantity (terms B and C) we can write the primitive perturbation series (7.32) as

$$\begin{aligned} Q(k) \ &= \ \left[G^{(0)}\right] w \left[G^{(0)}\right] + \lambda^2 G^{(0)} M Q^{(0)} Q^{(0)} M G^{(0)} \\ &\quad + \lambda^2 \left[G^{(0)} M G^{(0)} Q^{(0)} M G^{(0)}\right] w G^{(0)} \\ &\quad + \lambda^2 G^{(0)} w \left[G^{(0)} M G^{(0)} Q^{(0)} M G^{(0)}\right] + \mathcal{O}(\lambda^4), \end{aligned} \tag{7.36}$$

where the terms in square brackets are meant to represent quantities which behave or 'connect' (in the diagram language) as response functions or what Wyld calls in the language of field-theory 'propagators'. Wyld calls these propagator type terms 'reducible' and the ones which are left over are called 'irreducible'. Identifying terms in this way allows us to write (7.36) as

$$Q(k) = G w G + \lambda^2 G^{(0)} M Q^{(0)} Q^{(0)} M G^{(0)} + \mathcal{O}(\lambda^4), \tag{7.37}$$

where upon substitution of the **renormalized propagator** G given by

$$G = G^{(0)} + \lambda^2 G^{(0)} M G^{(0)} Q^{(0)} M G^{(0)} + \mathcal{O}(\lambda^4), \tag{7.38}$$

will yield our original primitive perturbation series given by (7.32) or equivalently (7.36).

An important fact to note here is that the exact or renormalized correlator is an observable whereas the zero-order or 'bare' correlator is not. This is analogous to the case in quantum field theory. However, in the case of the above formulation, the bare propagator is the observable whilst the renormalized propagator is not; contrary to the case in quantum field theory.

A similar renormalization procedure when applied to the M projectors, or what Wyld refers to as 'vertex' functions, from the rest of the irreducible term quantities provides us with a series for the renormalized vertex function. However, this and the rest of the renormalization procedure[2] will not be discussed here.

[2]This involves the further renormalization of terms in the series for the propagator and

One hopes that the brief explanation given above for the case of the renormalized propagator will provide a taste of the reasoning used to renormalize the rest of the terms in the primitive perturbation series. In summary the renormalization procedure has transformed our wildly divergent primitive series for the correlator, given by (7.32), to a renormalized series with unknown convergence properties given by

$$Q(k) = GwG + \lambda^2 G\mathbf{M}QQ\mathbf{M}G + \mathcal{O}(\lambda^4), \tag{7.39}$$

along with a renormalized series for the renormalized propagator

$$G = G^{(0)} + \lambda^2 G^{(0)} MGQMG + \mathcal{O}(\lambda^4), \tag{7.40}$$

and a renormalized series for the renormalized vertex function denoted by $\mathbf{M}$ and given by

$$\mathbf{M} = \lambda M + \lambda^3 \sum G\mathbf{M}Q\mathbf{M}G\mathbf{M} + \mathcal{O}(\lambda^5). \tag{7.41}$$

In summary and for the purposes of this thesis we can summarise this 'poor man's' explanation for the renormalization procedure as simply replacing the bare quantities with the exact or renormalized ones. The renormalized series are then truncated at some order in λ, and finally λ is set to unity.

With this in mind, it is now clearer why the QN approximation failed. Unlike the QN approximation, the above method can claim to be exact in the perturbation series and its renormalization. The role of the Gaussian field is as a bare field. The above procedure shows that the approach of going from Gaussian correlators to exact correlators has to be accompanied by going from exact observable propagators to renormalized propagators. Leaving the propagator unrenormalized, as the QN approximation does, will imply that the correlator can no-longer be regarded as exact. The vertex functions do not have to be renormalized for this to happen. In fact Wyld shows that a second-order truncation of (7.39), a zero-order truncation of (7.40) and a first order truncation of (7.41) returns Chandrasekhar's theory which is a time dependent generalisation of QN theory. McComb [5] mentions that Chandrasekhar's theory has in fact no renormalization at all and that the retention of the renormalization for the correlators is meaningless in the absence of any partial summations from the propagator. Wyld also shows that a second order truncation in (7.39) and

vertex functions for all orders.

(7.40) and a first order truncation in (7.41) results in Kraichnan's DIA, which will be discussed in the next section.

Lastly, one should also mention that Kraichnan (1977) [107] has also introduced an alternative method of justifying the renormalization procedure by using the method of reversion of power series.

7.4 Eulerian closures

7.4.1 The direct-interaction approximation (DIA)

The starting point of DIA is the introduction of an infinitesimal response tensor. Kraichnan studied the linear response of the NSE velocity field to an infinitesimal force perturbation

$$\begin{aligned}\left[\frac{\partial}{\partial t}+\nu k^2\right]\{u_\alpha(\mathbf{k},t)+\delta u_\alpha(\mathbf{k},t)\} &= \lambda M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\,\{u_\beta(\mathbf{j},t)u_\gamma(\mathbf{k}-\mathbf{j},t)\\ &\quad 2u_\beta(\mathbf{j},t)\delta u_\gamma(\mathbf{k}-\mathbf{j},t)+\delta u_\beta(\mathbf{j},t)\delta u_\gamma(\mathbf{k}-\mathbf{j},t)\}\\ &\quad +\{f_\alpha(\mathbf{k},t)+\delta f_\alpha(\mathbf{k},t)\}\,. \end{aligned} \tag{7.42}$$

Subtracting the NSE from this equation and keeping only the terms linear in δu results in the equation

$$\left[\frac{\partial}{\partial t}+\nu k^2\right]\delta u_\alpha(\mathbf{k},t)=\lambda M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j 2u_\beta(\mathbf{j},t)\delta u_\gamma(\mathbf{k}-\mathbf{j},t)+\delta f_\alpha(\mathbf{k},t). \tag{7.43}$$

Because only a linear response was considered, Kraichnan then wrote the solution to (7.43) in terms of a Greens tensor

$$\begin{aligned}&\left[\frac{\partial}{\partial t}+\nu k^2\right]\hat{G}_{\alpha\sigma}(\mathbf{k};t,t')-\lambda M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j 2u_\beta(\mathbf{j},t)\hat{G}_{\gamma\sigma}(\mathbf{k}-\mathbf{j};t,t')\\ &=P_{\alpha\sigma}(\mathbf{k})\delta\left(t-t'\right).\end{aligned} \tag{7.44}$$

Equation (7.44), along with the equation for the two-time covariance or correlation tensor

$$\left[\frac{\partial}{\partial t}+\nu k^2\right]Q_{\alpha\sigma}(\mathbf{k};t,t')=\lambda M_{\alpha\beta\gamma}\left(k\right)\int d^3j\,\left\langle u_\beta(\mathbf{j},t)u_\gamma(\mathbf{k}-\mathbf{j},t)u_\sigma(-\mathbf{k},t)\right\rangle, \tag{7.45}$$

can be solved perturbatively around a bare (viscous) response function and a bare (Gaussian) correlation function to yield

$$\begin{aligned}\left[\frac{\partial}{\partial t}+\nu k^2\right]Q_{\alpha\sigma}(\mathbf{k};t,t') &= \lambda^2 M_{\alpha\beta\gamma}(k)\int d^3j\left\{\left[\int_0^{t'} dsG^{(0)}_{\sigma\rho}(-\mathbf{k};t',s)\times\right.\right.\\ &\quad\left.\times\,2M_{\rho\delta\epsilon}(-\mathbf{k})Q^{(0)}_{\beta\delta}(\mathbf{j};t,s)Q^{(0)}_{\epsilon\gamma}(\mathbf{k}-\mathbf{j};t,s)\right]-\\ &\quad-\left[\int_0^{t} dsG^{(0)}_{\beta\rho}(\mathbf{j};t,s)M_{\rho\delta\epsilon}(\mathbf{j})\times\right.\\ &\quad\left.\left.\times 4Q^{(0)}_{\delta\gamma}(\mathbf{k}-\mathbf{j};t,s)Q^{(0)}_{\epsilon\sigma}(-\mathbf{k};t',s)\right]\right\}\\ &\quad+\mathcal{O}\left(\lambda^3\right), \end{aligned} \tag{7.46}$$

and

$$\begin{aligned}&\left[\frac{\partial}{\partial t}+\nu k^2\right]G_{\alpha\sigma}(\mathbf{k};t,t')\\ &-\left[4\lambda^2 M_{\alpha\beta\gamma}(k)\int d^3j\int_{t'}^{t} dsG^{(0)}_{\beta\rho}(\mathbf{j};t,s)M_{\rho\delta\epsilon}(\mathbf{j})\times\right.\\ &\quad\left.\times\,G^{(0)}_{\epsilon\sigma}(\mathbf{k};s,t')Q^{(0)}_{\gamma\delta}(\mathbf{k}-\mathbf{j};t,s)\right]+\mathcal{O}\left(\lambda^3\right)\\ &\quad=P_{\alpha\sigma}(\mathbf{k})\delta(t-t'), \end{aligned} \tag{7.47}$$

where the hat on the response function $\hat{G}_{\alpha\sigma}(\mathbf{k};t,t')$ in (7.44) has been removed due to intermediate averaging steps, which amount to a mean-field like approximation [108]. The process for the renormalization justified by the previous section can now be done by effectively replacing $Q^{(0)}$ by Q and $G^{(0)}$ by G in equations (7.46) and (7.47). The DIA derivation is completed by truncating both equations at second-order in λ and setting λ equal to unity. This truncation at second-order is what puts the 'direct interaction' in DIA. Kraichnan refers to triad-mode interactions such as $(\mathbf{k},\mathbf{j},\mathbf{k}-\mathbf{j})$ as direct interaction terms whilst terms which involve higher modal interactions are called 'indirect'. After some further simplification of reducing isotropic tensors to scalar form [5], the

final form of the DIA equation for the correlation and response functions are

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)Q\left(k;t,t'\right)$$
$$=\int d^3jL\left(\mathbf{k},\mathbf{j}\right)\left\{\int_0^{t'}dsG\left(k;t',s\right)Q\left(j;t,s\right)Q\left(\left|\mathbf{k}-\mathbf{j}\right|;t,s\right)\right.$$
$$\left.-\int_0^t dsG\left(j;t,s\right)Q\left(k;s,t'\right)Q\left(\left|\mathbf{k}-\mathbf{j}\right|;t,s\right)\right\}\ , \tag{7.48}$$

and

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)G\left(k;t,t'\right)$$
$$+\int d^3jL\left(\mathbf{k},\mathbf{j}\right)\int_{t'}^t dsG\left(j;t,s\right)G\left(k;s,t'\right)Q\left(\left|\mathbf{k}-\mathbf{j}\right|;t,s\right)$$
$$=\delta(t-t')\quad . \tag{7.49}$$

Unlike the QN theory DIA is physically realisable in terms of positive energy spectra. Although originally received with much acclaim, the DIA equations were later found to be incompatible with the Kolmogorov spectrum; DIA predicted a $k^{-3/2}$ power law spectrum in the inertial range compared with the Kolmogorov $k^{-5/3}$. Kraichnan attributed this failure of DIA to the theory not observing *random* Galilean invariance, a concept which he introduced. In turn he linked this to the theory being constructed in an Eulerian coordinate frame and thus suggested it should instead be cast in a Lagrangian frame.

In the Eulerian coordinate frame the velocity is described in terms of a 'field' i.e. the primary dependent variable is the velocity field $u_\alpha(\mathbf{x},t)$ which tells us the value of the fluid velocity at any position $\mathbf{x}$ and time t. One can also describe the fluid motion in a Lagrangian coordinate frame, where one tracks the paths made by a particular point of the fluid. Whereas in the Eulerian description the principal quantity is the field $u_\alpha(\mathbf{x},t)$, in the Lagrangian description it is the paths or *histories* which are of concern.

The motivation to fix DIA led to Kraichnan recasting DIA in quasi-Lagrangian coordinates which yielded the new 'Lagrangian-history direct interaction' (LHDI) and the later 'Abridged Lagrangian-history direct interaction' (ALHDI); which are compatible with the Kolmogorov spectrum. These in turn, however, did not give good answers for other things, which led to the development of fur-

ther extensions (and more acronyms) in the form of 'strain-based' ALHDI or SBALHDI.

7.4.2 Edwards-Fokker-Planck theory (EFP)

The EFP theory developed by Edwards [90, 109] was based on an analogy with the development of kinetic equations in statistical mechanics. Edwards started with the NSE cast in terms of a Liouville equation with the primary quantity of interest being the *time-independent* probability distribution functional of the velocity field realisations. Closed equations for the probability distribution functional were then obtained by approximating the Liouville equation to Fokker-Planck form by doing a perturbation expansion around a zero-order Gaussian distribution followed by a renormalization procedure different, but in a similar spirit, to the DIA and Wyld renormalization. The closed equations can then be used to construct an equation for the stationary correlation function which is given by

$$W(k) - 2\nu k^2 Q(k) = 2\int d^3j \frac{L(\mathbf{k},\mathbf{j})Q(|\mathbf{k}-\mathbf{j}|)\left[Q(k)-Q(j)\right]}{\omega(k)+\omega(j)+\omega(|\mathbf{k}-\mathbf{j}|)}\ , \tag{7.50}$$

where $W(k)$ is the energy input term and $\omega(k)$, which Edwards called the 'eddy diffusivity'[3], is given by

$$\omega(k) = \nu k^2 + \int d^3j \frac{L(\mathbf{k},\mathbf{j})Q(|\mathbf{k}-\mathbf{j}|)}{\omega(k)+\omega(j)+\omega(|\mathbf{k}-\mathbf{j}|)}\ , \tag{7.51}$$

and can be seen to be the equivalent of the response equation for EFP theory. EFP, like DIA, also suffers from incompatibility with the Kolmogorov spectrum. This takes the form of an infra-red divergence in the response equation (7.51), when the Kolmogorov spectrum is substituted as a solution for the energy spectrum i.e. $\omega(k) \to \infty$ as $k \to 0$. Although not mentioned earlier, this same analysis can also be applied to DIA resulting with an infra-red divergence in the response and reconfirming the incompatibility with the Kolmogorov spectrum.

[3]This should not to be confused with the earlier passive scalar eddy diffusivity.

7.4.3 Self-consistent field theory (SCF)

The SCF closure theory proposed by Herring [110] was originally cast in a similar language and procedure to EFP with a probability distribution functional determined by a Liouville equation. The formal renormalization procedure is more akin to the DIA in that we have a bare quantity which is then renormalized. However, the crucial element of SCF is the use of the 'self-consistent field' method which is its namesake. The self-consistent field method has a history of use in many-body quantum mechanics. In brief, and in the context of quantum mechanics, the self-consistent field method can be described as deriving a potential from a trial wavefunction for a system; the potential is then used in the Schrödinger equation to calculate the wavefunction for the system; self-consistency then requires that the original trial wavefunction and the subsequently calculated one are identical. Details of this method, as applied to the NSE by Herring, can be found in the original paper and also in [5]. Here we will simply quote the final equations in a form similar to the DIA and EFP theories above for comparison. The SCF equations are given by

$$\left(\frac{\partial}{\partial t}+\nu k^2\right) g\left(k;t,t'\right) = -\int d^3jL\left(\mathbf{k},\mathbf{j}\right)\int_{t'}^{t} dsg\left(j;t,s\right) g\left(k;s,t'\right) M\left(\left|\mathbf{k}-\mathbf{j}\right|;t,s\right) \ , \quad (7.52)$$

and

$$\left(\frac{\partial}{\partial t}+\nu k^2\right) M\left(k;t,t'\right) = -\int d^3jL\left(\mathbf{k},\mathbf{j}\right)\int_{t'}^{t} dsg\left(j;t,s\right) M\left(k;s,t'\right) M\left(\left|\mathbf{k}-\mathbf{j}\right|;t,s\right) \ , \quad (7.53)$$

where g can be seen to play a similar part to the DIA response function and M being the equivalent of the correlation function. Making this analogy also shows us that the response equations for DIA and SCF are identical; even with the differences of the derivation of each theory.

Lastly, looking at the two SCF equations, (7.52) and (7.53), one can also see from inspection that the following relationship holds between them

$$M\left(k;t,t'\right) = g\left(k;t,t'\right) M\left(k;t'\right) . \quad (7.54)$$

More on equations like this will be discussed in the following subsection and also in the next chapter where they form the fundamental equations for the LET theory of McComb.

7.4.4 Local energy transfer theory (LET)

We will be discussing McComb's LET theory in detail in the next chapter. However, a brief introduction will be made for the sake of completeness so that it can be compared with the other closure theories discussed above. LET was originally developed by McComb [111, 112, 113, 114] from Edward's EFP theory of stationary turbulence, as an attempt to cure the infra-red divergence of the response equation by demanding that energy conservation symmetries be met. McComb later rederived the LET theory of stationary turbulence [115] and then extended it to a time-dependent form [116, 117].

The latest derivation of LET [118] can be said to be based upon a DIA/Wyld renormalized perturbation scheme, where the relevant perturbation expansion is taken around a zero-order velocity field which is defined to have Gaussian statistics. Contrary to the DIA/Wyld schemes, the zero-order velocity field does not inherit its Gaussian nature from any forcing term; in the LET theory derivation, McComb does not include a forcing term in the NSE. In this sense the LET is based upon a free-decaying velocity field NSE rather than a forced one, although LET was later applied to the case of forced isotropic turbulence [108]. The LET equivalent of the DIA response function is the renormalized propagator H which links the velocity realisation at two different times in the following way

$$u_\alpha(\mathbf{k},t) = H_{\alpha\beta}\,(k;t,t')\,u_\beta(\mathbf{k},t'). \tag{7.55}$$

Equation (7.55) assumes that a velocity propagates from time t' to time t via a renormalized propagator. From the above ansatz of a renormalized propagator along with a mean-field assumption, was derived one of the main starting equations of the LET theory which in essence describes the behaviour of the renormalized propagator

$$Q(k;t,t') = H(k;t,t')Q(k;t',t'). \tag{7.56}$$

In microscopic equilibrium statistical mechanics one of the fundamental results

is a relation of this type known as a 'fluctuation-dissipation relation'. The use of such relations in macroscopic physics outside of equilibrium has been debated much in the latter half of the last century and this topic will be discussed in more detail in the next chapter. A comparison of (7.54) with (7.56) can also help to show the relation of LET and SCF.

Using (7.56) as a basis to calculate the renormalized propagator the final equations of LET theory are

$$\begin{aligned}
&\left(\frac{\partial}{\partial t}+\nu k^2\right) Q\left(k;t,t'\right) \\
&= \int d^3jL\left(\mathbf{k},\mathbf{j}\right)\left\{\int_0^{t'} dsH\left(k;t',s\right)Q\left(j;t,s\right)Q\left(\left|\mathbf{k}-\mathbf{j}\right|;t,s\right)\right. \\
&\left.-\int_0^{t} dsH\left(j;t,s\right)Q\left(k;s,t'\right)Q\left(\left|\mathbf{k}-\mathbf{j}\right|;t,s\right)\right\}
\end{aligned} \tag{7.57}$$

for the two-time correlator, and

$$\begin{aligned}
&\left(\frac{\partial}{\partial t}+\nu k^2\right) H\left(k;t,t'\right)+ \\
&+\int d^3jL\left(\mathbf{k},\mathbf{j}\right)\int_{t'}^{t} dsH\left(j;t,s\right)H\left(k;s,t'\right)Q\left(\left|\mathbf{k}-\mathbf{j}\right|;t,s\right) \\
&=\int d^3jL\left(\mathbf{k},\mathbf{j}\right)\int_0^{t'} ds\frac{Q\left(\left|\mathbf{k}-\mathbf{j}\right|;t,s\right)}{Q\left(k;t',t'\right)}\times \\
&\times\left\{H\left(k;t',s\right)Q\left(j;t,s\right)-H\left(j;t,s\right)Q\left(k;t',s\right)\right\}.
\end{aligned} \tag{7.58}$$

for the renormalized propagator. Comparison with the DIA response equation reveals the addition of the term on the LHS of (7.58). McComb *et. al.* claim that this term cancels the DIA infra-red divergence problems for when the Kolmogorov spectrum is introduced as a solution. Thus, it is claimed that because of this, the LET theory is compatible with the Kolmogorov spectrum and in turn is the **only** Eulerian closure to be compatible.

7.4.5 Overview of two-time closures

Looking over the various equations for the correlation and response/propagator functions one can see that they do not differ too significantly in form. In fact, all the theories bar EFP (which applies to stationary turbulence only) have the

same single-time correlation equation given by

$$\left(\frac{\partial}{\partial t}+2\nu k^2\right)Q\left(k;t\right)$$
$$=2\int d^3jL\left(\mathbf{k},\mathbf{j}\right)\int_0^t dsQ\left(\left|\mathbf{k}-\mathbf{j}\right|;t,s\right)\times$$
$$\left[H\left(k;t,s\right)Q\left(j;t,s\right)-H\left(j;t,s\right)Q\left(k;t,s\right)\right], \tag{7.59}$$

where we have used the notation of McComb's LET for the response/propagator H. When equation (7.59) is reduced to stationary form and on assumption of exponential forms for the correlation and response/propagator

$$H\left(k;t,s\right) = \exp\left\{-\omega(k)[t-s]\right\}, \tag{7.60}$$
$$Q\left(k;t,s\right) = Q(k)\exp\left\{-\omega(k)[t-s]\right\}, \tag{7.61}$$

first introduced by Kraichnan [119], we obtain the EFP equation for the stationary correlation function; thus showing that all the above theories result in the same stationary time-independent behaviour. Some might say that it is not entirely surprising that they are all very much similar as they are all effectively second order truncations of a renormalized perturbation series based on Gaussian zero-order solutions, in some way or another.

7.5 Single-time Markovianized closures

Even with the success (and failures) of the RPT's discussed above in closing the NSE moment hierarchy, the equations are still considered quite a challenge in computational terms; and this complaint is for the homogeneous, isotropic and incompressible case. As soon as one begins to introduce inhomogeneities as in the case for most simulations of practical use, the equations that need to be calculated become significantly more complex. With this in mind there is a need for computationally less intensive 'models'. By 'models' we mean theories which require some specific assumption which usually results in an adjustable free parameter normally fixed by experiment. Most real cases of turbulence simulation are of this kind with the most famous being the $k-\epsilon$ model which is the work-horse of most practical simulations [5]. With the failure of the QN theory and the development of RPT's, methods were suggested in

this direction. These resulted in models which took as their main assumption the concept of *Markovianization* which can be basically stated to be that in the evolution of the desired quantity in time, the quantity in question only depends on the preceding time step, but not on any step before that. A brief introduction to some of the more popular schemes is now given.

7.5.1 QN & EDQN

The equation for the correlation function from the quasi-normal approximation (as derived earlier) is

$$\begin{aligned}&\left(\frac{\partial}{\partial t}+2\nu k^2\right)Q\left(k;t\right)\\&=2\int d^3jL\left(\mathbf{k},\mathbf{j}\right)\int_0^t dse^{\left\{-\nu\left(k^2+j^2+|\mathbf{k}-\mathbf{j}|^2\right)(t-s)\right\}}\times\\&\times Q\left(\left|\mathbf{k}-\mathbf{j}\right|;s\right)\left[Q\left(j;s\right)-Q\left(k;s\right)\right].\end{aligned}\tag{7.62}$$

The primary failure of the QN approximation is that it is unrealisable i.e. it results in negative energy spectra. Orszag [87] suggested a remedy in the form of an added eddy-damping factor $\eta(k)$ which takes account of the missing memory loss in the system due to phase decoherence of the velocity Fourier modes. This was done by the substitution $\nu k^2 \rightarrow \nu k^2+\eta(k)$ so that (7.62) becomes the 'eddy damped quasi-normal (EDQN) approximation'

$$\begin{aligned}&\left(\frac{\partial}{\partial t}+2\nu k^2\right)Q\left(k;t\right)\\&=2\int d^3jL\left(\mathbf{k},\mathbf{j}\right)\int_0^t dse^{\{-(\omega(k)+\omega(j)+\omega(|\mathbf{k}-\mathbf{j}|))(t-s)\}}\times\\&\times Q\left(\left|\mathbf{k}-\mathbf{j}\right|;s\right)\left[Q\left(j;s\right)-Q\left(k;s\right)\right],\end{aligned}\tag{7.63}$$

where $\omega(k)=\nu k^2+\eta(k)$. All that remains is to ascertain an expression for $\eta(k)$. Various strategies have been proposed for this but we will just quote the most widely used choice [87, 120, 121, 84, 85]. This is based upon choosing the eddy damping $\eta(k)$ to be a dimensionally sound combination of local-in-k variables, where k exists in the inertial range

$$\eta(k)=\beta\epsilon^{\frac{1}{3}}k^{\frac{2}{3}},\tag{7.64}$$

where β is a dimensionless coefficient determined by experience [120], usually calculated by assuming an experimentally determined value of the Kolmogorov constant.

7.5.2 EDQNM

A further simplification can be made to EDQN via the step of Markovianization [120]. Lesieur [84] states that the inclusion of an eddy-damping term does not in itself guarantee realisability, and that Markovianization is needed for this. This simply involves updating the past values of $Q(s)$ to the current values of $Q(t)$. The Markovianization step allows a considerable simplification of the spectral equation and results in the 'eddy-damped quasi-normal Markovian (EDQNM)' single-time closure equations

$$\begin{aligned}&\left(\frac{\partial}{\partial t}+2\nu k^2\right)Q\left(k;t\right)\\&=2\int d^3jL\left(\mathbf{k},\mathbf{j}\right)\theta\left(k,j,\left|\mathbf{k}-\mathbf{j}\right|;t\right)\times\\&\times Q\left(\left|\mathbf{k}-\mathbf{j}\right|;t\right)\left[Q\left(j;t\right)-Q\left(k;t\right)\right],\end{aligned}\tag{7.65}$$

$$\theta\left(k,j,\left|\mathbf{k}-\mathbf{j}\right|;t\right)=\frac{1-\exp\left\{-\left[\omega(k)+\omega(j)+\omega\left(\left|\mathbf{k}-\mathbf{j}\right|\right)\right]t\right\}}{\omega(k)+\omega(j)+\omega\left(\left|\mathbf{k}-\mathbf{j}\right|\right)},\tag{7.66}$$

$$\omega(k)=\nu k^2+\beta\epsilon^{\frac{1}{3}}k^{\frac{2}{3}},\tag{7.67}$$

where the last equation for $\omega(k)$ varies depending on use, author etc., but like EDQN, seems to be the most widely used form; and the function $\theta\left(k,j,\left|\mathbf{k}-\mathbf{j}\right|;t\right)$ is called the memory-time function.

7.5.3 TFM

In the same paper, Orszag [87], suggested that a more fundamental measure of relaxation effects is the Green's function of the equations and not $\omega(k)$. This led to the treatment of eddy damping effects as given by Kraichnan's DIA.

Kraichnan's [82] 'test-field model (TFM)' is a single-time closure based on DIA but starting from a model Langevin-type equation. Details of the derivation

of the TFM can be found in [82, 122, 67, 123]. The TFM single-time closure begins by Markovianizing the eddy damping in the DIA Langevin equation. After the Markovian step, the following three dynamical equations [122] are derived

$$\begin{aligned}&\left(\frac{\partial}{\partial t}+2\nu k^2\right)Q\left(k;t\right)\\&=2\int d^3jL\left(\mathbf{k},\mathbf{j}\right)D(k,j,\left|\mathbf{k}-\mathbf{j}\right|;t)\times\\&\times Q\left(\left|\mathbf{k}-\mathbf{j}\right|;t\right)\left[Q\left(j;t\right)-Q\left(k;t\right)\right],\end{aligned}\tag{7.68}$$

$$\eta(k;t)=\int d^3jL\left(\mathbf{k},\mathbf{j}\right)D(k,j,\left|\mathbf{k}-\mathbf{j}\right|;t)Q\left(\left|\mathbf{k}-\mathbf{j}\right|;t\right),\tag{7.69}$$

$$\begin{aligned}\frac{\partial}{\partial t}D(k,j,\left|\mathbf{k}-\mathbf{j}\right|;t)&=1-\left[\left(\nu k^2+\nu j^2+\nu\left|\mathbf{k}-\mathbf{j}\right|^2\right)+\eta(k;t)+\eta(j;t)\right.\\&\left.+\eta\left(\left|\mathbf{k}-\mathbf{j}\right|;t\right)\right]D(k,j,\left|\mathbf{k}-\mathbf{j}\right|;t),\end{aligned}\tag{7.70}$$

where $D(k,j,\left|\mathbf{k}-\mathbf{j}\right|;t)$ is the memory-time function. We should note here that Kraichnan [82] does not write the first equation in this form; we have done this to facilitate later comparison. This expression is originally written as

$$\begin{aligned}&\left(\frac{\partial}{\partial t}+2\nu k^2+2\eta(k,t)\right)Q\left(k;t\right)\\&=2\int d^3jL\left(\mathbf{k},\mathbf{j}\right)D(k,j,\left|\mathbf{k}-\mathbf{j}\right|;t)Q\left(\left|\mathbf{k}-\mathbf{j}\right|;t\right)Q\left(j;t\right),\end{aligned}\tag{7.71}$$

from which it is now easier to see the origins of (7.69). The set of equations (7.68) - (7.70) constitute what Kraichnan calls the Generalised Edwards model as it extends EFP theory to time dependent form.

As with EFP the Generalised Edwards model is not compatible with K41. Thus, a further modification was made by Kraichnan [82] to take into account invariance to *random Galilean transformation* in a similar way that the Lagrangian-history direct-interaction approximation did with the DIA K41 incompatibility. Kraichnan did this by using the interaction between solenoidal and compressive parts of a test-field to better determine the memory-time function. This further complication introduces three extra equations to take into account the seperate solenoidal and compressive Green's functions of the test-field intro-

duced in the integrals. The form of the TFM equations along with more details of the derivation can be found in [82].

Again, as in the case of EDQNM, TFM depends upon the fixing of an adjustable constant. This takes the form of a scaling factor which ensures that the results of some values (e.g. the Kolmogorov constant) predicted by TFM agree with the Lagrangian-history direct-interaction approximation.

Chapter 8

Local energy transfer theory

In this chapter we will be revising previous derivations of the local energy transfer (LET) theory. In doing this, some problems with previous derivations will be highlighted and thus we will aim to provide some solutions to these problem so as to make the theory more consistent. Particular attention has been given to making the LET renormalized propagator manifestly causal by making use of the Heaviside unit-step function. This was done so that the symmetries of the various relations involved are preserved. Inconsistencies in the development of previous LET theories were thus removed. New consistent representations of the correlation function, or correlator, are presented in which the concept of time-ordering is introduced to conserve the symmetry of the correlator. Various essential properties of the propagator, such as transitivity, are proved and the compatibility of LET with the Kolmogorov spectrum is shown. The equation of motion for the LET propagator is re-derived and as a result, now contains a counter-term removing the singularity of previous propagator equations at $t = t'$. We will finish this chapter by deriving a new single-time Markovianized model from the LET theory.

8.1 The LET ansatz

8.1.1 Problems with the previous (1992) derivation

The previous derivation of the LET theory by McComb *et al.* (1992) [118] started with the ansatz

$$u_\alpha(\mathbf{k},t) = H_{\alpha\sigma}(\mathbf{k};t,s)\,u_\sigma(\mathbf{k},s)\,, \tag{8.1}$$

where $H_{\alpha\sigma}(\mathbf{k};t,s)$ is the postulated renormalized propagator linking velocities at two different times. From equation (8.1) and the assumption that the propagator is statistically sharp, $\langle H\rangle = H$, was derived (see Appendix C.1) the most important of the LET equations

$$Q(k;t,t') = \theta(t-t')\,H(k;t,t')\,Q(k;t',t')\,, \tag{8.2}$$

where the Heaviside unit-step function $\theta(t-t')$ defined as

$$\begin{aligned} \forall\ \tau \geq 0 \qquad \theta(\tau) &= 1, \\ \forall\ \tau < 0 \qquad \theta(\tau) &= 0, \end{aligned} \tag{8.3}$$

indicates the causal nature of the propagator.

The correlator's time-reversal symmetry requires that

$$Q(k;t,t') = Q(k;t',t)\,. \tag{8.4}$$

This result follows from isotropy (see Appendix C.2). Using (8.2) to expand both sides of (8.4) we find the contradiction

$$\theta(t-t')\,H(k;t,t')\,Q(k;t',t') = \theta(t'-t)\,H(k;t',t)\,Q(k;t,t)\,. \tag{8.5}$$

Equation (8.5) implies that unless $t = t'$, we must have $Q(k;t,t') = 0 = Q(k;t',t)$. This in turn implies that all the equations of the LET theory trivially become zero. This contradiction in the theory originates from (8.2). The LHS of (8.2) is time-reversal symmetric whereas the RHS is clearly not. This in turn arises from **not** explicitly declaring the time-ordering involved. In the LET literature, the time-ordering if explicitly declared appears at the beginning

of the formulation and then is implicitly assumed.

The following subsections reassess the formulation of the LET ansatz (8.2) and sort out the problem of time-symmetries involved by making extensive use of the Heaviside unit-step function to show time-ordering explicitly.

8.1.2 Formulation of the LET ansatz

We begin by considering the generalised covariance equation

$$\left[\frac{\partial}{\partial t}+\nu k^2\right] Q_{\alpha\sigma}\left(\mathbf{k};t,t'\right)=\lambda M_{\alpha\beta\gamma}\left(\mathbf{k}\right)\int d^3j\left\langle u_\beta\left(\mathbf{j},t\right)u_\gamma\left(\mathbf{k}-\mathbf{j},t\right)u_\sigma\left(-\mathbf{k},t'\right)\right\rangle, \tag{8.6}$$

where as in earlier chapters λ is a book-keeping parameter which is set to unity at the end of the perturbative calculation. By using an integrating factor and integrating over time we can write this as

$$\begin{aligned} Q_{\alpha\sigma}\left(\mathbf{k};t,t'\right) &= H^{(0)}_{\alpha\varepsilon}\left(\mathbf{k};t,s\right)Q_{\epsilon\sigma}\left(\mathbf{k};s,t'\right)+ \\ &\quad+\left[\lambda\int_s^t dt''H^{(0)}_{\alpha\epsilon}\left(\mathbf{k};t,t''\right)M_{\epsilon\beta\gamma}\left(\mathbf{k}\right)\times\right. \\ &\quad\left.\times\int d^3j\left\langle u_\beta\left(\mathbf{j},t''\right)u_\gamma\left(\mathbf{k}-\mathbf{j},t''\right)u_\sigma\left(-\mathbf{k},t'\right)\right\rangle\right], \end{aligned} \tag{8.7}$$

where the integrating factor

$$H^{(0)}_{\alpha\epsilon}\left(\mathbf{k};t,t''\right)=\begin{cases} P_{\alpha\epsilon}\left(\mathbf{k}\right)e^{-\nu k^2(t-t'')} & t\geq t'', \\ 0 & t<t'' \end{cases} \tag{8.8}$$

If on both sides of (8.7), we expand the u's which comprise $Q_{\alpha\sigma}\left(\mathbf{k};t,t'\right)$, in a perturbation series around a Gaussian velocity field[1]

$$Q_{\alpha\sigma}\left(\mathbf{k};t,t'\right)=Q^{(0)}_{\alpha\sigma}\left(\mathbf{k};t,t'\right)+\lambda^2Q^{(2)}_{\alpha\sigma}\left(\mathbf{k};t,t'\right)\cdots, \tag{8.9}$$

we can see that $H^{(0)}_{\alpha\epsilon}\left(\mathbf{k};t,s\right)$ acts as a zero-order propagator for the zero-order correlator

$$Q^{(0)}_{\alpha\sigma}\left(\mathbf{k};t,t'\right)=\theta\left(t-s\right)H^{(0)}_{\alpha\epsilon}\left(\mathbf{k};t,s\right)Q^{(0)}_{\epsilon\sigma}\left(\mathbf{k};s,t'\right). \tag{8.10}$$

[1] This implies that all odd order moments are zero.

This is an **exact** result. We shall call this the zero-order or ***bare*** result.

Re-arranging (8.7) to prompt the next step

$$\begin{aligned} & Q_{\alpha\sigma}\left(\mathbf{k};t,t'\right) \\ & = \left[H_{\alpha\varepsilon}^{(0)}\left(\mathbf{k};t,s\right)+\frac{1}{Q_{\epsilon\sigma}\left(\mathbf{k};s,t'\right)}\lambda\int_{s}^{t}dt''H_{\alpha\epsilon}^{(0)}\left(\mathbf{k};t,t''\right)M_{\epsilon\beta\gamma}\left(\mathbf{k}\right)\times\right. \\ & \left.\times\int d^{3}j\left\langle u_{\beta}\left(\mathbf{j},t''\right)u_{\gamma}\left(\mathbf{k}-\mathbf{j},t''\right)u_{\sigma}\left(-\mathbf{k},t'\right)\right\rangle\right]Q_{\epsilon\sigma}\left(\mathbf{k};s,t'\right), \end{aligned} \tag{8.11}$$

we postulate that we may write this in its **renormalized** form as

$$Q_{\alpha\sigma}\left(\mathbf{k};t,t'\right)=\theta\left(t-s\right)H_{\alpha\epsilon}\left(\mathbf{k};t,s\right)Q_{\epsilon\sigma}\left(\mathbf{k};s,t'\right), \tag{8.12}$$

or by using (7.9), in its isotropic version as

$$Q\left(k;t,t'\right)=\theta\left(t-s\right)H\left(k;t,s\right)Q\left(k;s,t'\right). \tag{8.13}$$

We have effectively replaced the zero-order equation (8.10) by its renormalized version using the replacements

$$\begin{aligned} Q^{(0)} & \rightarrow Q, \\ H^{(0)} & \rightarrow H. \end{aligned} \tag{8.14}$$

The $\theta(t-s)$ in (8.13) is explicitly informing us of the **causality condition**. This basically tells us that no future times, $s>t$, can affect the correlation of two velocities at times t and t' respectively. The renormalized propagator (8.13) can be said to propagate a correlation between two velocities at times s and t' to a correlation between two velocities at times t and t'.

As yet we have no information about the time-ordering of the two times t and t', and thus the symmetry under interchange of t and t' holds in (8.13). If we explicitly state the time ordering as $t>t'$ say, then this is equivalent to applying the Heaviside unit-step function $\theta(t-t')$ to both sides of (8.13)

$$\theta\left(t-t'\right)Q\left(k;t,t'\right)=\theta\left(t-t'\right)\theta\left(t-s\right)H\left(k;t,s\right)Q\left(k;s,t'\right), \tag{8.15}$$

and this is the beginning of the LET theory. In it, we have postulated the existence of a renormalized propagator and have made extensive use of the

Heaviside unit-step function to make the time-ordering manifest.

8.1.3 The generalised fluctuation-dissipation relation

By setting $s = t'$ in (8.15) we obtain the important relation

$$\theta\left(t-t'\right)Q\left(k;t,t'\right)=\theta\left(t-t'\right)H\left(k;t,t'\right)Q\left(k;t',t'\right)\ . \tag{8.16}$$

Comparing this with equation (8.2) we can easily see that the problem of the incorrect time symmetries has been overcome simply by including the missing step of making the time-ordering manifest by the presence of the unit-step function on the LHS of (8.16). The result in equation (8.16) takes the form of a fluctuation-dissipation relationship (or FDR). Such relationships which are most familiar in microscopic systems at thermal equilibrium [124], have over the past three decades been discussed in the context of the way in which relationships like this occur in turbulence theory ([125, 126] and references therein). More recently Frederiksen and Davies [127] have distinguished between spectral renormalized perturbation theories by the way in which relationships of the form of (8.16) play a part. Also there has been a considerable amount of interest in the existence of FDR's in macroscopic non-equilibrium physics from a dynamical systems point of view [128, 129, 130, 131], and also in condensed matter in which systems relaxing to equilibrium have been studied and shown to exhibit such generalised FDR's [132].

8.1.4 Symmetric representation of the correlator

We now introduce a representation of the correlator which preserves the symmetry under interchange of time arguments, in the form

$$Q\left(k;t,t'\right)=\theta\left(t-t'\right)Q\left(k;t,t'\right)+\theta\left(t'-t\right)Q\left(k;t,t'\right)-\delta_{t,t'}Q\left(k;t,t'\right). \tag{8.17}$$

We can show that this representation does what it is supposed to do by looking at the separate cases:-
Case 1 $t > t'$

$$\begin{aligned}\theta(t-t')\,Q(k;t,t') &= \theta(t-t')\,\theta(t-t')\,Q(k;t,t') + \\ &\quad +\theta(t-t')\,\theta(t'-t)\,Q(k;t,t') \\ &\quad -\theta(t-t')\,\delta_{t,t'}Q(k;t,t') \\ &= \theta(t-t')\,Q(k;t,t') + \delta_{t,t'}Q(k;t,t') - \delta_{t,t'}Q(k;t,t') \\ &= \theta(t-t')\,Q(k;t,t') \qquad (8.18)\end{aligned}$$

Case 2 $t < t'$

Similar method to the above case shows

$$\theta(t'-t)\,Q(k;t,t') = \theta(t'-t)\,Q(k;t,t') \qquad (8.19)$$

Case 3 $t = t'$

$$\begin{aligned}\delta_{t,t'}Q(k;t,t') &= \delta_{t,t'}\theta(t-t')\,Q(k;t,t') + \\ &\quad +\delta_{t,t'}\theta(t'-t)\,Q(k;t,t') \\ &\quad -\delta_{t,t'}\delta_{t,t'}Q(k;t,t') \\ &= \delta_{t,t'}Q(k;t,t') + \delta_{t,t'}Q(k;t,t') - \delta_{t,t'}Q(k;t,t') \\ &= \delta_{t,t'}Q(k;t,t') \qquad (8.20)\end{aligned}$$

Using (8.15) to expand the RHS of (8.17) we obtain

$$\begin{aligned}Q(k;t,t') &= \theta(t-t')\,\theta(t-s)\,H(k;t,s)\,Q(k;s,t') \\ &\quad +\theta(t'-t)\,\theta(t'-p)\,H(k;t',p)\,Q(k;p,t) \\ &\quad -\delta_{t,t'}Q(k;t,t')\,. \qquad (8.21)\end{aligned}$$

Equation (8.21) may be written more like the FDR by using (8.16) to construct it instead

$$\begin{aligned} Q\left(k;t,t'\right) &= \theta\left(t-t'\right)H\left(k;t,t'\right)Q\left(k;t',t'\right) \\ &\quad +\theta\left(t'-t\right)H\left(k;t',t\right)Q\left(k;t,t\right) \\ &\quad -\delta_{t,t'}Q\left(k;t,t'\right). \end{aligned} \tag{8.22}$$

The symmetry of both these correlators, (8.21) and (8.22), can be broken by applying a unit-step function to both sides. This will yield something like (8.15) or (8.16) depending on what time-ordering one chooses. This completes the treatment for the three cases.

As shown in Leslie (1973) [81], a representation for the 2-time correlator of stationary turbulence, first introduced by Kraichnan, is given by

$$Q(k;t-t') = Q(k)exp\left\{-\omega(k)\left|t-t'\right|\right\}, \tag{8.23}$$

where the modulus is needed in the exponential to ensure (8.4), and where $Q(k)$ is the stationary correlator or spectral density function and $\omega(k)$ is the total eddy-decay rate. The advantage of the new representation (8.22), along with the propagator given by the assumed form

$$H(k;t,t') = \theta(t-t')exp\left\{-\omega(k)(t-t')\right\}, \tag{8.24}$$

is that this representation has the property that

$$\lim_{t\to t'}\frac{\partial}{\partial t}Q(k;t,t') = 0, \tag{8.25}$$

as it should for stationary turbulence (see Appendix C.3 for detailed proof). In contrast to this, the use of (8.23) fails to obtain this result (see Leslie [81] p.93). This is a result of the (8.22) representation exhibiting the time-reversal symmetry $t \leftrightarrow t'$ in a more manifest way than (8.23). Leslie, for his calculation, only takes $t > t'$ for the representation of the correlator, whereas we have determined in (8.4) that the correlator is symmetric under interchange of t and t'. Also, (8.22) is compatible with the highly desirable exponential representation of the correlator which fails in the (8.23) representation due to the above reason.

8.2 Properties of the Propagator

The renormalized propagator can be shown to exhibit certain properties which can be proved by use of the symmetric representation introduced earlier. Many of these properties e.g. transitivity of the propagator, have previously been assumed without proof [118, 108, 133].

The simple property of the propagator

$$H(k;t,t) = 1, \tag{8.26}$$

can be easily shown to be necessary by setting $s = t$ in (8.15).

Another set of properties can be obtained by equating the RHS of (8.15) with the RHS of (8.16)

$$\begin{aligned} \theta(t-t')\,H(k;t,t')\,Q(k;t',t') &= \theta(t-t')\,\theta(t-s)\times \\ &\quad \times H(k;t,s)\,Q(k;s,t')\,. \end{aligned} \tag{8.27}$$

Expanding the RHS of (8.27) using (8.22) we get

$$\begin{aligned} &\theta(t-t')\,H(k;t,t')\,Q(k;t',t') \\ &\left.\begin{aligned} &= [\theta(t-t')\,\theta(t-s)\,H(k;t,s)\times \\ &\times\theta(s-t')\,H(k;s,t')\,Q(k;t',t')] \end{aligned}\right\} a \\ &\left.\begin{aligned} &+ [\theta(t-t')\,\theta(t-s)\,H(k;t,s)\times \\ &\times\,\theta(t'-s)\,H(k;t',s)\,Q(k;s,s)] \end{aligned}\right\} b \\ &\left.\begin{aligned} &- [\theta(t-t')\,\theta(t-s)\,H(k;t,s) \\ &\qquad \times\,\delta_{t',s}Q(k;s,t')]\,. \end{aligned}\right\} c \end{aligned} \tag{8.28}$$

Dividing the RHS into three groups of terms labeled respectively a,b and c, we will now look at (8.28) for three separate cases:-

Case 1 $t > s > t'$: **Propagator Transitivity**
This corresponds to $b = 0$ and $c = 0$, leaving

$$\begin{aligned} &\theta(t-t')\,H(k;t,t')\,Q(k;t',t') \\ &= [\theta(t-t')\,\theta(t-s)\,H(k;t,s)\times \\ &\times\theta(s-t')\,H(k;s,t')\,Q(k;t',t')]\,. \end{aligned} \tag{8.29}$$

We now use the contraction property of the Heaviside function (see Appendix C) $\theta(t-s)\,\theta(s-t') = \theta(t-t')$ to write (8.29) as

$$\begin{aligned} &\theta(t-t')\,\underline{H(k;t,t')}Q(k;t',t') \\ &= \theta(t-t')\,\underline{H(k;t,s)\,H(k;s,t')}Q(k;t',t') \end{aligned} \tag{8.30}$$

From this above result, we can deduce the **transitive property** of the propagator

$$H(k;t,t') = H(k;t,s)\,H(k;s,t')\,. \tag{8.31}$$

This result also tells us that the transitivity property of the propagator holds only for times s which are *intermediate* between the two times t and t'. This makes sense because otherwise, if s was outside the range between t and t', you would have propagation backwards in time which violates causality. One should note that, as mentioned earlier, this is a result which was previously assumed in the LET literature.

Case 2 $t \geq t' > s$: **Linking/Propagating single-time correlators**
This corresponds to $a = 0$ and $c = 0$, leaving

$$\begin{aligned} &\theta(t-t')\,H(k;t,t')\,\underline{Q(k;t',t')} \\ &= \big[\theta(t-t')\,\theta(t-s)\,H(k;t,s) \\ &\times \theta(t'-s)\,H(k;t',s)\,\underline{Q(k;s,s)}\big] \end{aligned} \tag{8.32}$$

This result is important because it links two single-time correlators. This fact becomes clearer if we take the special case of $t = t'$. This leaves

$$Q(k;t,t) = \theta(t-s)\,H(k;t,s)\,H(k;t,s)\,Q(k;s,s)\,, \tag{8.33}$$

implying that we need two propagators to link single-time correlators. Defining

$$\widetilde{H}(k;t,s) := H(k;t,s)\,H(k;t,s)\,, \tag{8.34}$$

equation (8.33) can be modified to make it look like (8.13)

$$Q(k;t) = \theta(t-s)\,\widetilde{H}(k;t,s)\,Q(k;s)\,. \tag{8.35}$$

Again the unit-step function is simply telling us that correlators, and hence the

energy spectrum from equation (1.25), can only propagate forwards in time. An interesting point to note here is that if we had started the whole LET RPT scheme with the single-time covariance instead of the 2-time form, we would have naturally obtained relation (8.35) as the equivalent to the relation (8.16). The fact that we have two propagators in relation (8.35) simply follows from noticing that in the single-time covariance equation we have the dissipation term containing $2\nu k^2$ instead of the νk^2 which the 2-time covariance equation (8.6) contains.

Case 3 $t \geq t' = s$
Lastly, and for the sake of completeness, this case results in

$$\theta\left(t-t'\right) H\left(k;t,t'\right) Q\left(k;t',t'\right) = \theta\left(t-t'\right) H\left(k;t,t'\right) Q\left(k;t',t'\right). \tag{8.36}$$

8.3 Derivation of the LET renormalized correlator equation

The LET correlator is derived in a similar way to the DIA correlation function. We follow the Wyld perturbation scheme introduced in the previous chapter, and start by substituting the perturbation series of $u_\alpha(\mathbf{k},t)$ around a Gaussian velocity field $u_\alpha^{(0)}(\mathbf{k},t)$

$$u_\alpha(\mathbf{k},t) = u_\alpha^{(0)}(\mathbf{k},t) + \lambda u_\alpha^{(1)}(\mathbf{k},t) + \lambda^2 u_\alpha^{(2)}(\mathbf{k},t) + \mathcal{O}\left(\lambda^3\right) \tag{8.37}$$

into equation (8.6) for the covariance. Equating terms of the same order in λ allows one to determine the coefficients $u_\alpha^{(n)}(\mathbf{k},t)$ in terms of the zero-order Gaussian velocity field $u_\alpha^{(0)}(\mathbf{k},t)$. Substituting these coefficients back into the expression for the covariance, and evaluating the Gaussian zero-order mo-

ments in terms of $Q^{(0)}$ results in

$$\left[\frac{\partial}{\partial t}+\nu k^2\right]Q_{\alpha\sigma}(\mathbf{k};t,t') = \lambda^2 M_{\alpha\beta\gamma}(k)\int d^3j\left\{\left[\int_0^{t'} dsH^{(0)}_{\sigma\rho}(-\mathbf{k};t',s)M_{\rho\delta\epsilon}(-\mathbf{k})\times \times 2Q^{(0)}_{\beta\delta}(\mathbf{j};t,s)Q^{(0)}_{\epsilon\gamma}(\mathbf{k}-\mathbf{j};t,s)\right]- - \left[\int_0^{t} dsH^{(0)}_{\beta\rho}(\mathbf{j};t,s)M_{\rho\delta\epsilon}(\mathbf{j})\times \times 4Q^{(0)}_{\delta\gamma}(\mathbf{k}-\mathbf{j};t,s)Q^{(0)}_{\epsilon\sigma}(-\mathbf{k};t',s)\right]\right\}+\mathcal{O}\left(\lambda^3\right), \quad (8.38)$$

where $H^{(0)}$ is the zero-order bare viscous propagator. We now wave the 'renormalization wand' and implement the Kraichnan/Wyld renormalization recipe, outlined in the previous chapter, as follows.

1. Make the replacements

$$\begin{aligned} Q^{(0)} &\rightarrow Q, \\ H^{(0)} &\rightarrow H. \end{aligned} \quad (8.39)$$

2. Truncate the renormalized expansion at second order in λ and set $\lambda = 1$.

3. Specialise to the case of isotropic turbulence and reduce all second-rank isotropic tensors to scalar form by using (7.7).

With this done we obtain the LET correlator equation

$$\begin{aligned} &\left(\frac{\partial}{\partial t}+\nu k^2\right)Q\left(k;t,t'\right) \\ &= \int d^3jL\left(\mathbf{k},\mathbf{j}\right)\left\{\int_0^{t'} dsH\left(k;t',s\right)Q\left(j;t,s\right)Q\left(\left|\mathbf{k}-\mathbf{j}\right|;t,s\right)\right. \\ &\left.-\int_0^{t} dsH\left(j;t,s\right)Q\left(k;s,t'\right)Q\left(\left|\mathbf{k}-\mathbf{j}\right|;t,s\right)\right\}. \end{aligned} \quad (8.40)$$

which, not surprisingly, is identical to the DIA version or equivalently, a second-order truncation in a Wyld renormalized perturbation series.

8.4 Derivation of the LET propagator equation

The significant departure of LET from other spectral RPT's comes in the description of its response or propagator equation, the basis of which is equation (8.16). To derive the evolution equation for the LET propagator we can now proceed in two ways.

1. The first is to substitute (8.22) in (8.40) and then choose $t > t'$.
2. The second is to choose $t > t'$ and thus multiply both sides of (8.40) by $\theta\left(t-t'\right)$ to show the range over which the equation will be valid. Then follow this by using (8.16) throughout.

Both methods are equivalent but the second is the easiest to use in practice. Thus we begin by choosing the time-ordering to be $t \geq t'$ and multiplying (8.40) by $\theta(t-t')$

$$\begin{aligned}
&\theta(t-t')\frac{\partial}{\partial t}Q\left(k;t,t'\right)+\theta(t-t')\nu k^2 Q\left(k;t,t'\right)\\
&=\theta(t-t')\int d^3jL\left(\mathbf{k},\mathbf{j}\right)\left\{\int_0^{t'}dsH\left(k;t',s\right)Q\left(j;t,s\right)Q\left(\left|\mathbf{k}-\mathbf{j}\right|;t,s\right)\right.\\
&\left.-\int_0^{t}dsH\left(j;t,s\right)Q\left(k;s,t'\right)Q\left(\left|\mathbf{k}-\mathbf{j}\right|;t,s\right)\right\}. \qquad (8.41)
\end{aligned}$$

Let us look at the first term of the LHS of (8.41)

$$\begin{aligned}
&\theta\left(t-t'\right)\frac{\partial}{\partial t}Q\left(k;t,t'\right)\\
&=\frac{\partial}{\partial t}\theta\left(t-t'\right)Q\left(k;t,t'\right)-Q\left(k;t,t'\right)\frac{\partial}{\partial t}\theta\left(t-t'\right)\\
&=\frac{\partial}{\partial t}\theta\left(t-t'\right)H\left(k;t,t'\right)Q\left(k;t',t'\right)\\
&\quad -Q\left(k;t,t'\right)\frac{\partial}{\partial t}\theta\left(t-t'\right), \qquad (8.42)
\end{aligned}$$

where we have applied the product rule in the 2^{nd} line, and have substituted (8.16) in the 3^{rd} line. After substituting the differential of the Heaviside unit-step function

$$\frac{\partial}{\partial t}\theta\left(t-t'\right)=\delta(t-t'), \qquad (8.43)$$

we get to our final form for this part of the propagator equation. So we find that the LHS of (8.41) has been reduced to

$$\begin{aligned} &LHS\ of\ (8.41) \\ &= \frac{\partial}{\partial t}\theta\left(t-t'\right)H\left(k;t,t'\right)Q\left(k;t',t'\right) - Q\left(k;t,t'\right)\delta(t-t') \\ &+\nu k^2\theta\left(t-t'\right)H\left(k;t,t'\right)Q\left(k;t',t'\right), \end{aligned} \tag{8.44}$$

where (8.16) was used on the second term of the LHS of (8.41) also.

We now evaluate the 2^{nd} time integral on the RHS of (8.41) , which we will label as TI_2

$$TI_2 = \theta\left(t-t'\right)\int_0^t dsH\left(j;t,s\right)Q\left(k;s,t'\right)Q\left(\left|\mathbf{k}-\mathbf{j}\right|;t,s\right). \tag{8.45}$$

We need to have the appropriate θ functions in front of the correlators so that the broken time-reversal symmetry becomes manifest. This information is present in the arguments of the propagator and in $\theta\left(t-t'\right)$. So for the $Q\left(\left|\mathbf{k}-\mathbf{j}\right|;t,s\right)$ correlator we have

$$\begin{aligned} \theta\left(t-t'\right)\int_0^t dsQ\left(\left|\mathbf{k}-\mathbf{j}\right|;t,s\right) &= \theta\left(t-t'\right)\int_0^t ds\theta\left(t-s\right)\times \\ &\quad \times Q\left(\left|\mathbf{k}-\mathbf{j}\right|;t,s\right), \end{aligned} \tag{8.46}$$

and for the $Q\left(k;s,t'\right)$ correlator we have

$$\begin{aligned} \theta\left(t-t'\right)\int_0^t dsQ\left(k;s,t'\right) &= \theta\left(t-t'\right)\int_0^{t'} dsQ\left(k;s,t'\right) \\ &+ \theta\left(t-t'\right)\int_{t'}^t dsQ\left(k;s,t'\right) \\ &= \theta\left(t-t'\right)\int_0^{t'} ds\theta\left(t'-s\right)Q\left(k;t',s\right) \\ &+ \theta\left(t-t'\right)\int_{t'}^t ds\theta\left(s-t'\right)Q\left(k;s,t'\right), \end{aligned} \tag{8.47}$$

where we have used (8.4) in the fourth line. With these results we can now write (8.45) as

$$\begin{aligned} & TI_2 \\ & = \left[\theta\left(t-t'\right)\int_{t'}^{t} dsH\left(j;t,s\right)\theta\left(s-t'\right)\times\right. \\ & \quad \left.\times\, Q\left(k;s,t'\right)\theta\left(t-s\right)Q\left(\left|\mathbf{k}-\mathbf{j}\right|;t,s\right)\right] \\ & + \left[\theta\left(t-t'\right)\int_{0}^{t'} dsH\left(j;t,s\right)\theta\left(t'-s\right)\times\right. \\ & \quad \left.\times\, Q\left(k;t',s\right)\theta\left(t-s\right)Q\left(\left|\mathbf{k}-\mathbf{j}\right|;t,s\right)\right]. \end{aligned} \tag{8.48}$$

The evaluation of the first integral on the RHS of (8.41) follows similarly so that the final LET response equation is given by

$$\begin{aligned} & \frac{\partial}{\partial t}\theta\left(t-t'\right)H\left(k;t,t'\right)Q\left(k;t',t'\right)-Q\left(k;t,t'\right)\delta\left(t-t'\right) \\ & +\nu k^2\theta\left(t-t'\right)H\left(k;t,t'\right)Q\left(k;t',t'\right) \\ = & \int d^3jL\left(\mathbf{k},\mathbf{j}\right)\theta\left(t-t'\right)\times \\ & \times\left\{\int_{0}^{t'} dsH\left(k;t',s\right)\theta\left(t-s\right)Q\left(j;t,s\right)\theta\left(t-s\right)Q\left(\left|\mathbf{k}-\mathbf{j}\right|;t,s\right)\right. \\ & -\int_{0}^{t'} dsH\left(j;t,s\right)\theta\left(t'-s\right)Q\left(k;t',s\right)\theta\left(t-s\right)Q\left(\left|\mathbf{k}-\mathbf{j}\right|;t,s\right) \\ & -\left[\int_{t'}^{t} dsH\left(j;t,s\right)\theta\left(s-t'\right)\times\right. \\ & \quad \left.\left.\times\, Q\left(k;s,t'\right)\theta\left(t-s\right)Q\left(\left|\mathbf{k}-\mathbf{j}\right|;t,s\right)\right]\right\}. \end{aligned} \tag{8.49}$$

Multiplying both sides by $\theta(t-t')$, dividing by $Q\left(k;t',t'\right)$ and realising that (8.16) implies

$$\frac{\theta(t-t')Q\left(k;t,t'\right)}{Q\left(k;t',t'\right)}=\theta(t-t')H(k;t,t'), \tag{8.50}$$

we get to the simplified form with the broken time-reversal symmetry manifest

$$\theta(t-t')\left(\frac{\partial}{\partial t}+\nu k^2\right)\theta(t-t')\,H(k;t,t') - \theta(t-t')H(k;t,t')\delta(t-t') + \int d^3jL(\mathbf{k},\mathbf{j})\,\theta(t-t')\int_{t'}^{t} dsH(j;t,s)\,H(k;s,t')\,\theta(t-s)\,Q(|\mathbf{k}-\mathbf{j}|;t,s) = \int d^3jL(\mathbf{k},\mathbf{j})\,\theta(t-t')\int_0^{t'} ds\frac{\theta(t-s)\,Q(|\mathbf{k}-\mathbf{j}|;t,s)}{Q(k;t',t')}\times \times\left\{H(k;t',s)\,\theta(t-s)\,Q(j;t,s) - H(j;t,s)\,\theta(t'-s)\,Q(k;t',s)\right\}. \quad (8.51)$$

Apart from the addition of the second term on the LHS

$$-\theta(t-t')H(k;t,t')\delta(t-t'), \quad (8.52)$$

(8.51) is the same as equation (3.19) in [118], equation (20) in [108] and equation (7.146) in [5]. The natural addition of this extra term as a consequence of time-ordering, fixes the problem of the singularity in the time-derivative of the response equation (8.51) which occurs when one takes $t = t'$.

8.5 The LET Equations

In summary the revised equations are now

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)Q(k;t,t') = \int d^3jL(\mathbf{k},\mathbf{j})\left\{\int_0^{t'} dsH(k;t',s)\,Q(j;t,s)\,Q(|\mathbf{k}-\mathbf{j}|;t,s) - \int_0^{t} dsH(j;t,s)\,Q(k;s,t')\,Q(|\mathbf{k}-\mathbf{j}|;t,s)\right\}, \quad (8.53)$$

for the two-time correlation (off-diagonal),

$$\left(\frac{\partial}{\partial t}+2\nu k^2\right)Q(k;t) = 2\int d^3jL(\mathbf{k},\mathbf{j})\int_0^{t} dsH(k;t,s)\,H(j;t,s)\,H(|\mathbf{k}-\mathbf{j}|;t,s)\times [Q(j;s)\,Q(|\mathbf{k}-\mathbf{j}|;s) - Q(k;s)\,Q(|\mathbf{k}-\mathbf{j}|;s)], \quad (8.54)$$

for the one-time correlation (on-diagonal)[2] and

$$\theta(t-t')\left(\frac{\partial}{\partial t}+\nu k^2\right)\theta(t-t')H(k;t,t')-\theta(t-t')H(k;t,t')\delta(t-t')$$
$$+\int d^3jL(\mathbf{k},\mathbf{j})\theta(t-t')\int_{t'}^{t}dsH(j;t,s)H(k;s,t')\theta(t-s)Q(|\mathbf{k}-\mathbf{j}|;t,s)$$
$$=\int d^3jL(\mathbf{k},\mathbf{j})\theta(t-t')\int_{0}^{t'}ds\frac{\theta(t-s)Q(|\mathbf{k}-\mathbf{j}|;t,s)}{Q(k;t',t')}\times$$
$$\times\{H(k;t',s)\theta(t-s)Q(j;t,s)-H(j;t,s)\theta(t'-s)Q(k;t',s)\}, \qquad (8.55)$$

for the LET propagator function. The above equations along with the generalised fluctuation-dissipation relation (FDR),

$$\theta(t-t')Q(k;t,t')=\theta(t-t')H(k;t,t')Q(k;t'), \qquad (8.56)$$

from which LET is derived, and the single-time correlator link equation

$$Q(k;t)=\theta(t-s)H(k;t,s)H(k;t,s)Q(k;s), \qquad (8.57)$$

complete the set of important LET equations.

One should note that equation (8.54) is the normal equation for the spectral function with the FDR (8.56) applied to it, so that all two-time correlators are turned into one-time correlators.

8.5.1 Partial propagator representation

We may write the propagator in a representation which separates the discontinuous part as a unit-step function.

$$H(k;t,t')=\theta(t-t')\mathcal{H}(k;t,t'), \qquad (8.58)$$

where $\mathcal{H}(k;t,t')$ is a representation of the propagator BUT without the discontinuity at $t=t'$. So using (8.58) and the FDR (8.56) to turn two-time correlators

[2] Derived in a similar renormalization scheme to the two-time correlator.

into single-time form, (8.55) becomes:

$$\begin{aligned}
&\theta\left(t-t'\right)\left(\frac{\partial}{\partial t}+\nu k^{2}\right)\mathcal{H}\left(k;t,t'\right)\\
&=-\theta\left(t-t'\right)\int d^{3}jL\left(\mathbf{k},\mathbf{j}\right)\int_{t'}^{t}ds\left\{\mathcal{H}\left(k;s,t'\right)\times\right.\\
&\left.\times\mathcal{H}\left(j;t,s\right)\mathcal{H}\left(\left|\mathbf{k}-\mathbf{j}\right|;t,s\right)Q\left(\left|\mathbf{k}-\mathbf{j}\right|;s\right)\right\}\\
&+\theta\left(t-t'\right)\int d^{3}jL\left(\mathbf{k},\mathbf{j}\right)\int_{0}^{t'}ds\left\{\mathcal{H}\left(k;t',s\right)\mathcal{H}\left(j;t,s\right)\mathcal{H}\left(\left|\mathbf{k}-\mathbf{j}\right|;t,s\right)\times\right.\\
&\left.\times\frac{Q\left(\left|\mathbf{k}-\mathbf{j}\right|;s\right)}{Q\left(k;t'\right)}\left[Q\left(j;s\right)-Q\left(k;s\right)\right]\right\},
\end{aligned}\tag{8.59}$$

the propagator equation for $t\geq t'$. The counter-term has been canceled by use of the product rule in the time-derivative.

8.5.2 Behaviour of the LET response in the limit of infinite Reynolds number

The later constructions of the LET theory claimed that their solutions were compatible with the Kolmogorov spectrum. However, this was not ever shown explicitly. Compatibility will now be shown for the LET response/propagator as given by (8.59). This will follow a similar procedure given by McComb (1990) [5], and simply amounts to the requirement that the eddy-decay rate term, $\omega(k)$, of the respective theory be consistent with the limiting behaviour of the infinite Reynolds number Kolmogorov type form for the eddy-decay rate given by $\omega(k)=\beta\varepsilon^{1/3}k^{2/3}$. An IR divergence in $\omega(k)$, as is the case with DIA, EFP etc., is clearly not consistent.

We begin by writing (8.59) in stationary form. This simply means that all single-time correlators become time independent

$$Q(k;t)\rightarrow Q(k),\tag{8.60}$$

and we write the propagator in relative/difference time coordinates

$$\mathcal{H}\left(k;t,t'\right)=\mathcal{H}\left(k;t-t'\right).\tag{8.61}$$

Next we assume the exponential form for the propagator as introduced earlier

$$\mathcal{H}\left(k;t-t'\right)=\exp[-\omega(k)(t-t')], \tag{8.62}$$

where, as before, $\omega(k)$ is the total eddy-decay rate. These changes result in the propagator equation

$$\begin{aligned}
&\theta\left(t-t'\right)\left(\frac{\partial}{\partial t}+\nu k^2\right)\exp[-\omega(k)(t-t')]\\
&=\left[-\theta\left(t-t'\right)\int d^3jL\left(\mathbf{k},\mathbf{j}\right)Q\left(|\mathbf{k}-\mathbf{j}|\right)\times\right.\\
&\left.\times\int_{t'}^{t}ds\exp[-\omega(k)(s-t')-\omega(j)(t-s)-\omega(|\mathbf{k}-\mathbf{j}|)(t-s)]\right]\\
&+\left[\theta\left(t-t'\right)\int d^3jL\left(\mathbf{k},\mathbf{j}\right)\times\right.\\
&\times\int_{0}^{t'}ds\left\{\exp[-\omega(k)(t'-s)-\omega(j)(t-s)-\omega(|\mathbf{k}-\mathbf{j}|)(t-s)]\times\right.\\
&\left.\left.\times\frac{Q\left(|\mathbf{k}-\mathbf{j}|\right)}{Q\left(k\right)}\left[Q\left(j\right)-Q\left(k\right)\right]\right\}\right].
\end{aligned} \tag{8.63}$$

Doing the differentiation, setting $t=t'$ and finally calculating the time integration results in an equation for $\omega(k)$

$$\begin{aligned}
\omega(k) &= \nu k^2+\left\{\int d^3jL\left(\mathbf{k},\mathbf{j}\right)\frac{Q\left(|\mathbf{k}-\mathbf{j}|\right)\left[Q\left(j\right)-Q\left(k\right)\right]}{Q\left(k\right)\left[\omega(k)+\omega(j)+\omega(|\mathbf{k}-\mathbf{j}|)\right]}\times\right.\\
&\left.\times\left(1-\exp[-(\omega(k)+\omega(j)+\omega(|\mathbf{k}-\mathbf{j}|))t]\right)\right\},
\end{aligned} \tag{8.64}$$

where one can ignore the last term involving the exponential as we are considering stationary systems which are time independent. The simplest way to think of the neglecting of this term is to realise that since we are considering stationary cases only, we can take $t\to\infty$ and thus the exponential term will vanish in this limit. This residual term is a result of our choice of initial time conditions; if we had chosen $t=-\infty$ instead of $t=0$, this term would not have arisen.

To show that (8.64) is not divergent we complete our analysis by substituting the inertial range/ infinite Reynolds number forms for $Q(k)$ and $\omega(k)$ [5] given by

$$Q(k)=\frac{\alpha\varepsilon^{2/3}}{4\pi}k^{-11/3}, \tag{8.65}$$

$$\omega(k) = \beta\varepsilon^{1/3}k^{2/3}, \tag{8.66}$$

and by writing the integral in k, j, μ variables as in the previous chapter on RG

$$\begin{aligned}\omega(k) &= \nu k^2 + \left\{ \int dj \int d\mu \frac{kj^3(\mu^2-1)[\mu(k^2+j^2) - kj(1+2\mu^2)]}{k^2+j^2-2kj\mu} \times \right. \\ &\left. \times \frac{\alpha\beta^{-1}\varepsilon^{1/3} \left|\mathbf{k}-\mathbf{j}\right|^{-11/3} \left[j^{-11/3} - k^{-11/3}\right]}{k^{-11/3}\left[k^{2/3}+j^{2/3}+\left|\mathbf{k}-\mathbf{j}\right|^{2/3}\right]} \right\}\end{aligned} \tag{8.67}$$

where μ is the cosine of the angle between the two vectors $\mathbf{k}$ and $\mathbf{j}$.

There are three possible sources of divergence in this expression. From equation (8.67) it can be seen that the $k \to 0$ and $j \to 0$ limits do not pose a problem. The final possible source $|\mathbf{k}-\mathbf{j}| \to 0$ can be resolved by realising that the term $\left[j^{-11/3} - k^{-11/3}\right]$ cancels the divergence caused by the $|\mathbf{k}-\mathbf{j}|^{-11/3}$ term. This is shown by expanding

$$|\mathbf{k}-\mathbf{j}|^{-11/3} = \left(k^2+j^2-2kj\mu\right)^{-11/6}, \tag{8.68}$$

and substituting in equation (8.67). One then Taylor expands k around j to leading order in $\epsilon = k - j$ in both the numerator and denominator of the integrand in equation (8.67). This results in the integrand becoming

$$\frac{(11/3)\alpha\beta^{-1}\varepsilon^{1/3}}{(2)^{17/6}} \frac{[2j^2\mu(1-\mu) - j^2]}{j^{16/6}\left[2j^{2/3} + (2j^2)^{1/3}(1-\mu)^{1/3}\right]} (\mu^2-1)(1-\mu)^{-17/6}\epsilon\,, \tag{8.69}$$

and focusing on the term $(\mu^2-1)(1-\mu)^{-17/6}\epsilon$ we can see that as $\epsilon \to 0$, the integrand goes to zero, except at $\mu = 1$ where the integrand is singular. This singularity can be avoided if we write the limits of the μ integral as

$$\int_{-1}^{1} d\mu \to \int_{-1}^{\uparrow 1} d\mu\,, \tag{8.70}$$

where $\uparrow 1$ implies in the limit approaching 1 from below.

This completes the analysis in the limit of infinite Reynolds number. Further information on the above technique can be found in [5].

8.6 A single-time Markovianized LET closure

We will now attempt to construct a single-time theory based upon LET, akin to the various Markovianized theories discussed earlier in the previous chapter. One should emphasise that the motivation to Markovianize is only justified by the existence of these other models such as EDQNM and TFM. We will show that unlike these other single-time models, the LET based single-time Markovianized model does not have any free parameters to fix; it only relies on the assumption of Markovianization.

The relevant single time LET equations are the single-time correlator (using partial propagator form)

$$\begin{aligned}
&\left(\frac{\partial}{\partial t}+2\nu k^2\right)Q\left(k;t\right)\\
&=2\int d^3jL\left(\mathbf{k},\mathbf{j}\right)\int_0^t ds\mathcal{H}\left(k;t,s\right)\mathcal{H}\left(j;t,s\right)\mathcal{H}\left(\left|\mathbf{k}-\mathbf{j}\right|;t,s\right)\times\\
&\left[Q\left(j;s\right)Q\left(\left|\mathbf{k}-\mathbf{j}\right|;s\right)-Q\left(k;s\right)Q\left(\left|\mathbf{k}-\mathbf{j}\right|;s\right)\right],
\end{aligned}\tag{8.71}$$

the propagator equation

$$\begin{aligned}
&\theta\left(t-t'\right)\left(\frac{\partial}{\partial t}+\nu k^2\right)\mathcal{H}\left(k;t,t'\right)\\
&=-\theta\left(t-t'\right)\int d^3jL\left(\mathbf{k},\mathbf{j}\right)\int_{t'}^t ds\left\{\mathcal{H}\left(k;s,t'\right)\times\right.\\
&\left.\times\mathcal{H}\left(j;t,s\right)\mathcal{H}\left(\left|\mathbf{k}-\mathbf{j}\right|;t,s\right)Q\left(\left|\mathbf{k}-\mathbf{j}\right|;s\right)\right\}\\
&+\theta\left(t-t'\right)\int d^3jL\left(\mathbf{k},\mathbf{j}\right)\int_0^{t'} ds\left\{\mathcal{H}\left(k;t',s\right)\mathcal{H}\left(j;t,s\right)\mathcal{H}\left(\left|\mathbf{k}-\mathbf{j}\right|;t,s\right)\times\right.\\
&\left.\times\frac{Q\left(\left|\mathbf{k}-\mathbf{j}\right|;s\right)}{Q\left(k;t'\right)}\left[Q\left(j;s\right)-Q\left(k;s\right)\right]\right\},
\end{aligned}\tag{8.72}$$

and the single-time correlator link equation

$$Q(k;t)=\theta(t-s)H(k;t,s)H(k;t,s)Q(k;s),\tag{8.73}$$

Making a Markovian approximation we can write (8.71) as:

$$\begin{aligned}
\left(\frac{\partial}{\partial t}+2\nu k^2\right)Q\left(k;t\right) &= 2\int d^3jL\left(\mathbf{k},\mathbf{j}\right)D(k,j,\left|\mathbf{k}-\mathbf{j}\right|;t)\times\\
&\times Q\left(\left|\mathbf{k}-\mathbf{j}\right|;t\right)\left[Q\left(j;t\right)-Q\left(k;t\right)\right],
\end{aligned}\tag{8.74}$$

where the Markovian approximation amounts to updating the $Q(s)'s$ to $Q(t)'s$, and where

$$D(k,j,|\mathbf{k}-\mathbf{j}|;t)=\int_0^t ds\mathcal{H}(k;t,s)\,\mathcal{H}(j;t,s)\,\mathcal{H}(|\mathbf{k}-\mathbf{j}|;t,s)\,, \tag{8.75}$$

is the **memory time**.

We now need some way of computing $D(k,j,|\mathbf{k}-\mathbf{j}|;t)$ i.e. of updating it. We do this by differentiating (8.75) w.r.t. t to get

$$\begin{aligned}\frac{\partial}{\partial t}D(k,j,|\mathbf{k}-\mathbf{j}|;t) \;=1\;&+\int_0^t ds\left[\left(\frac{\partial}{\partial t}\mathcal{H}(k;t,s)\right)\mathcal{H}(j;t,s)\,\mathcal{H}(|\mathbf{k}-\mathbf{j}|;t,s)\right.\\&+\mathcal{H}(k;t,s)\left(\frac{\partial}{\partial t}\mathcal{H}(j;t,s)\right)\mathcal{H}(|\mathbf{k}-\mathbf{j}|;t,s)\\&\left.+\mathcal{H}(k;t,s)\,\mathcal{H}(j;t,s)\left(\frac{\partial}{\partial t}\mathcal{H}(|\mathbf{k}-\mathbf{j}|;t,s)\right)\right]. \end{aligned}\tag{8.76}$$

To evaluate (8.76) we need to know the dynamical behaviour of $\mathcal{H}(k;t,s)$ etc. We get this from (8.72). Proceed by writing (8.72) in a Langevin equation type form

$$\theta(t-t')\left[\frac{\partial}{\partial t}+\nu k^2+\eta(k;t,t')\right]\mathcal{H}(k;t,t')=0, \tag{8.77}$$

where

$$\begin{aligned}&\eta(k;t,t')\\&=\theta(t-t')\int d^3jL(\mathbf{k},\mathbf{j})\int_{t'}^t ds\,\{\mathcal{H}(k;s,t')\times\\&\times\mathcal{H}(j;t,s)\,\mathcal{H}(|\mathbf{k}-\mathbf{j}|;t,s)\,Q(|\mathbf{k}-\mathbf{j}|;s)\}\\&-\theta(t-t')\int d^3jL(\mathbf{k},\mathbf{j})\int_0^{t'} ds\left\{\mathcal{H}(k;t',s)\,\mathcal{H}(j;t,s)\,\mathcal{H}(|\mathbf{k}-\mathbf{j}|;t,s)\times\right.\\&\left.\times\frac{Q(|\mathbf{k}-\mathbf{j}|;s)}{Q(k;t')}\left[Q(j;s)-Q(k;s)\right]\right\},\end{aligned}\tag{8.78}$$

is the turbulent eddy-decay rate (or eddy-damping term; depending on literature) and is obtained by comparing with (8.72). Rearranging (8.77) we get

$$\theta(t-t')\frac{\partial}{\partial t}\mathcal{H}(k;t,t')=-\theta(t-t')\left[\nu k^2+\eta(k;t,t')\right]\mathcal{H}(k;t,t')\,. \tag{8.79}$$

Equation (8.79) allows us to write (8.76) as

$$\begin{aligned}\frac{\partial}{\partial t}D(k,j,|\mathbf{k}-\mathbf{j}|;t) &= 1- \int_0^t ds\left[\mathcal{H}(k;t,s)\,\mathcal{H}(j;t,s)\,\mathcal{H}(|\mathbf{k}-\mathbf{j}|;t,s)\times\right.\\ &\quad\times\left[\left(\nu k^2+\nu j^2+\nu|\mathbf{k}-\mathbf{j}|^2\right)+\eta(k;t,s)\right.\\ &\quad\left.\left.+\eta(j;t,s)+\eta(|\mathbf{k}-\mathbf{j}|;t,s)\right]\right].\end{aligned} \tag{8.80}$$

To be able to calculate (8.80) we will need to make the Markovian step

$$\eta(k;t,s)\rightarrow\eta(k;t). \tag{8.81}$$

We can justify this step by looking at equations (8.73), (8.74) and (8.77). Equation (8.77) gives a general solution

$$\mathcal{H}(k;t,t')=\exp\left\{-\nu k^2(t-t')-\int_{t'}^t ds\eta(k;s,t')\right\}. \tag{8.82}$$

If we write (8.74) in the suggestive form

$$\left[\frac{\partial}{\partial t}+2\nu k^2+2\xi(k;t)\right]Q(k;t)=0, \tag{8.83}$$

where

$$\xi(k;t)=-\int d^3jL(\mathbf{k},\mathbf{j})\,D(k,j,|\mathbf{k}-\mathbf{j}|;t)\frac{Q(|\mathbf{k}-\mathbf{j}|;t)}{Q(k;t)}\left[Q(j;t)-Q(k;t)\right], \tag{8.84}$$

then we can write the general solution of (8.83) as

$$\begin{aligned}Q(k;t) &= \exp\left\{-2\nu k^2(t-t')-2\int_{t'}^t ds\xi(k;s)\right\}Q(k;t')\\ &= \left[\exp\left\{-\nu k^2(t-t')-\int_{t'}^t ds\xi(k;s)\right\}\right]^2 Q(k;t').\end{aligned} \tag{8.85}$$

Writing (8.73) as

$$Q(k;t)=\theta(t-t')\mathcal{H}(k;t,t')\mathcal{H}(k;t,t')Q(k;t'), \tag{8.86}$$

and comparing with (8.85), this suggests that

$$\mathcal{H}(k;t,t')=\exp\left\{-\nu k^2(t-t')-\int_{t'}^t ds\xi(k;s)\right\}. \tag{8.87}$$

However, comparing this with (8.82) we see that

$$\int_{t'}^{t} ds\xi(k;s) = \int_{t'}^{t} ds\eta(k;s,t'). \tag{8.88}$$

Comparing the forms of $\xi(k;s)$ and $\eta(k;t,s)$, (8.84) and (8.78), we see that provided we make the consistent step of Markovianizing the correlators in equation (8.78) in the same way as how (8.74) was formed, we will get the result

$$\xi(k;s) = \eta(k;s,s). \tag{8.89}$$

Also in (8.88), since both t and t' are arbitrary such that we can make $t \sim t'$, we may make the important assumption that

$$\begin{aligned} ds\eta(k;s,t') &= ds\xi(k;s) \\ \eta(k;s,t') &= \xi(k;s) \\ &= \eta(k;s,s) = \eta(k;s). \end{aligned} \tag{8.90}$$

This tells us that in the case of the $\eta(k;s,t')$ term, we need only concern ourselves with the on-diagonal terms[3] $\eta(k;s,s) = \eta(k;s)$, which is a result of a Markovian simplification.

Going back to (8.80), we can now write it as

$$\begin{aligned} \frac{\partial}{\partial t} D(k,j,|\mathbf{k}-\mathbf{j}|;t) &= 1 - \Big[\left(\nu k^2 + \nu j^2 + \nu\left|\mathbf{k}-\mathbf{j}\right|^2\right) + \eta(k;t) + \eta(j;t) \\ &\quad + \eta\left(|\mathbf{k}-\mathbf{j}|;t\right)\Big] D(k,j,|\mathbf{k}-\mathbf{j}|;t), \end{aligned} \tag{8.91}$$

which along with (8.74) can be used to evolve the memory time.

[3]Note that Leslie (1973) in deriving an equation for $\eta(k;t)$ from DIA, averages over the second time argument i.e. $\eta(k;t) = \int_0^t ds\eta(k;t,s)$, whereas we simply take the on-diagonal terms. In effect Leslie's $\eta(k;t)$ should be written as $\overline{\eta}(k;t)$ showing that it is an averaged quantity.

8.6.1 Single-time LET equations

So the final equations for the single-time evolution are now

$$\left(\frac{\partial}{\partial t}+2\nu k^2\right)Q\left(k;t\right) = 2\int d^3jL\left(\mathbf{k},\mathbf{j}\right)D(k,j,\left|\mathbf{k}-\mathbf{j}\right|;t)\times \\ \times Q\left(\left|\mathbf{k}-\mathbf{j}\right|;t\right)\left[Q\left(j;t\right)-Q\left(k;t\right)\right] \\ = -2\eta(k;t)Q\left(k;t\right), \tag{8.92}$$

$$\eta(k;t)=-\int d^3jL\left(\mathbf{k},\mathbf{j}\right)D(k,j,\left|\mathbf{k}-\mathbf{j}\right|;t)\frac{Q\left(\left|\mathbf{k}-\mathbf{j}\right|;t\right)}{Q\left(k;t\right)}\left[Q\left(j;t\right)-Q\left(k;t\right)\right], \tag{8.93}$$

and

$$\frac{\partial}{\partial t}D(k,j,\left|\mathbf{k}-\mathbf{j}\right|;t) = 1-\Big[\left(\nu k^2+\nu j^2+\nu\left|\mathbf{k}-\mathbf{j}\right|^2\right)+\eta(k;t)+\eta(j;t) \\ +\eta\left(\left|\mathbf{k}-\mathbf{j}\right|;t\right)\Big]D(k,j,\left|\mathbf{k}-\mathbf{j}\right|;t). \tag{8.94}$$

These equations can be solved numerically with some initial conditions

$$Q(k;t=0) = \frac{E(k;t=0)}{4\pi k^2}, \tag{8.95}$$

where $E(k;t=0)$ is some initial energy spectra chosen, and

$$D(k,j,\left|\mathbf{k}-\mathbf{j}\right|;t=0)=0. \tag{8.96}$$

The last of these initial conditions follows from the definition of $D(k,j,\left|\mathbf{k}-\mathbf{j}\right|;t)$ in equation (8.75) and this in turn implies, from (8.93), that $\eta(k;t=0)=0$, as is expected because the cascade has not yet begun at $t=0$.

8.7 Conclusion

Earlier Eulerian spectral closures (DIA, EFP, SCF) suffered from infra-red divergences of their associated response functions in the limit of infinite Reynolds number. Thus their respective solutions were incompatible with the Kolmogorov spectrum for the inertial range. The earlier construction of LET by McComb

(1974) was built on Edwards' theory for stationary isotropic turbulence, and unlike the latter, was compatible with the Kolmogorov spectrum. Over the past three decades the LET theory was extended to two-time form, and its derivation increased in rigour over the course of a series of publications. Over this time, however the later form of the theory, although similar in name, was quite distinct from the derivation of the original 1974 paper which is more justified in name to be called 'local energy transfer theory'. Despite this, the name adhered to later theories. The current derivation provided above, improves upon previous theories by paying more attention to time-ordering and thus making all the divergences and singularities more manifest e.g. by the presence of the counter-term in the LET propagator equation. This apparentness allows us to deal with any unphysical inconsistencies in an easier way than if they were hidden. We have also improved upon earlier derivations by dealing directly with statistical quantities such as the correlation function, from the very outset where the LET ansatz was developed. Along with being more prudent, this also allows one to do away with any mean-field approximations of the older derivations. Lastly, the development of a single-time Markovianized model from the LET theory, has resulted in a set of closed equations which do not depend on any adjustable parameters; this being the main criticism leveled against models such as EDQNM and TFM.

Chapter 9

Conclusion

Through the course of this thesis we have aimed to obtain a better and more critical understanding of the two main renormalization schemes that are concerned with the statistical theory of isotropic turbulence. In particular, this study was primarily concerned with identifying and tackling some new and open problems of the renormalization schemes of McComb *et. al.*

The first of these methods was the Renormalization Group (RG) which looks for scale invariant behaviour by implementing small renormalization steps that enhance the transport variables in the NSE. After a brief review of the dynamical RG method and the attempts at applying this to the study of turbulence, we moved onto studying the RG method of McComb *et. al.* as presented in its latest form by McComb and Johnston [64, 96]. In Chapter 3 we recast the conditional average of the latter study into a functional based formalism. The derivation of this new conditional average projector involved the introduction of an operation which constricted the full ensemble, and an operation which then approximated this new conditioned ensemble by a representative ensemble in the subgrid band of k- space.

In Chapter 4 we presented the full RG calculation and corrected an error in a previous calculation of the effects of the time-history integral. We found that the correction introduced a new lifetime in the denominator of the increment to the viscosity. From this we moved onto Chapter 5 to identify some of the more serious problems of this RG scheme. This primarily involved identifying the importance of the cross-term interaction as the major contributor to the energy flux across the high/low-k partition if the partitioning wavenumber is in

the dissipation range. A simple kinetics based model was then heuristically constructed to see what qualitative effects the cross-term has on the renormalized viscosity. The model agreed qualitatively well with previous studies in this area, primarily that of numerical studies. However, due to the crudeness of the model, it possessed some inherent problems of its own.

In Chapter 6, we applied the RG study of McComb *et. al.* (with no cross-term effects) to the study of the scale invariant properties of a passive scalar field advected by a turbulent velocity field obeying the NSE.

The last part of this thesis is concerned with the other renormalization method which concerns the partial summing of terms in a bare primitive perturbation series into another renormalized series. These schemes are known as renormalized perturbation theories (RPT) and differ from the RG schemes by the fact that they include the effects of all modes on each other, rather than just the effects of the high-k modes on the low-k ones. These RPT schemes are very popular in turbulence research as they provide a closed model representation of the NSE, which is much easier to deal with than the NSE itself.

After a review of the different Eulerian RPT's, we moved onto providing, in Chapter 8, a new formulation of the local energy transfer (LET) theory of McComb. This study provides derivations of many useful results of which the most important is the introduction of time-ordering to fix some previous issues with handling the time-reversal symmetry of the correlator in conjunction with the time-irreversal nature of the propagator. We conclude this part of the thesis with the derivation of a new single-time Markovianized model based on LET theory, which unlike other theories of this kind does not rely on any arbitrary parameters.

Outlook on prospective work

The study presented in this thesis, although presenting some new ideas, has also provided insight into the *potential* of the schemes studied as well as highlighting older pending problems. We will conclude this thesis with a brief summary of some suggestions for future work.

RG

- In Chapter 5 we talked about the inclusion of higher order non-linearities in the RG procedure. Implicitly related to this is our truncation at second order in the local Reynolds number (effective coupling). Now although we know that the local Reynolds number is kept small, we do not know, however, the behaviour of the higher-order non-linearities in the convolutions. One feels that work is needed to determine the behaviour of such non-linearities in the RG iteration i.e. an analysis to determine if such terms involve relevant, irrelevant or marginal scaling fields.

- We also mentioned the idea of taking the RG calculation into a Lagrangian description, where the fluid is described in terms of paths/histories, so that we can retain some sort of phase information for each coarse-grained picture. In principle, one feels that by retaining higher-order non-linearities like the cubic term, we could do this to a certain extent in the Eulerian picture also. However, if we were to do this, we would still be needing a way of bolting these extra renormalized interactions onto a transport parameter at the end of the iteration. This also opens up the possibility of renormalizing the forcing and/or the vertex function parameters.

- To facilitate the above requirements, the necessary role of computation in working hand in hand with such schemes needs to be emphasised. In particular it would be interesting to see the physics of the system at each coarse-grained picture in the RG iteration. This would also help us to see how efficiently we are renormalizing certain key quantities e.g. the dissipation rate ε.

RPT

- Although quite formidable a task, some analysis is needed on the convergence properties of the partially summed renormalized perturbation series. Although, nearly every RPT scheme in turbulence follows a similar procedure to Kraichnan's DIA of truncating to second order (the 'direct' bit in the DIA), there are no assurances that this series converges or is even asymptotic. This is quite different to the case in quantum field theory (QFT) where the effects of higher order terms are shown to become less

and less important; in turbulence theory, it is *hoped* that the renormalized series do.

- Although quite elegant, one feels that RPT methods in turbulence theory still need to mature in comparison with fields such as QFT. For example, one thing to look at would be the use of the Gaussian zero order / bare velocity field. It is not hard to see that some of the structure of the RPT's discussed in Chapter 7 are in fact symptomatic of this bare field i.e. odd order moments becoming zero. This was also seen in the context of RG in Chapter 2, where Yakhot & Orszag had to resort to the construct of a 'correspondence principle' to evade this. A physical theory should be invariant to our choice of a bare field. In a similar sense to the use of RG in QFT, there is a need to invoke some sort of renormalization invariance to combat this.

- Finally as a response to the above points, it is a reassuring thing to know that we (in turbulence theory) are not alone and that many of these techniques have actually been developed to a considerable degree in other fields (QFT, critical phenomena) so that the necessary guidance, but not necessarily the complete solution, is there; even if it is implicit and needs a translation into turbulence theory. There is a need to bring together in a more unified framework some of the various facets of turbulence research mentioned at the beginning of this thesis. One would like to think that this is a reassuring and optimistic thought to look forward to; and it presents a well needed and worthwhile avenue for future studies.

Appendix A

Functionals and generalised functions

Earlier in Chapter 3 on the functional representation of the conditional average, we encountered some constructs and theorems pertaining to the use of functionals. Although implicit, we also dealt with probability distribution functions, and in Chapter 8 we used the Heaviside unit-step function. Both these functions are a particular form of functional known as generalised functions. In this appendix we will briefly review some important techniques and terminology on the theory of functionals and generalised functions that were used in this thesis. The notes here are a summary of those presented in Beran [69], Lumley [134], Binney *et. al* [135], Beerands *et. al.* [70] and also in the excellent online material by Torralba at the Universidad Complutense Madrid - Biophysics Theory `http://bbm1.ucm.es/torralba/funder/theory/`.

A.1 Theory of Functionals

A function can be considered to be a group of rules for mapping a set of numbers (the arguments of the function) to another number. For example the function $f(x, y) = x^2 \ln |y|$ is a function which takes two numbers (real or complex) and maps them to a single number; the function $g(x_1, x_2 \ldots x_n) = \sum_{i=1}^{n} \sqrt{x_i}$ takes n numbers and maps them to a single number etc.

A functional is the continuum version of a multi-variable function such as the ones described above. It takes as its argument a function for which it assigns,

via a set of rules for a mapping, a number. A simple example of a functional is

$$I[f(x)] = \int_b^a \ln f(x).dx \ , \tag{A.1}$$

where the argument of the functional is indicated by the contents of the square brackets, and where the mapping rule is simply to integrate the logarithm of the function $f(x)$ between the limits a and b. Thus we can see that a functional denoted by $I[f(x)]$ is a quantity that depends *continuously* upon all the values a function $f(x)$ takes in some interval $[a, b]$. Also a more important distinguishing factor between functions and functionals that this example illustrates, is that in the former x is not a dummy variable, whereas in the latter it quite clearly is. This is why, that in some cases, functionals like the one above would be denoted by $I[f]$ instead of $I[f(x)]$. Another simple example of a functional, this time without the use of integration, is

$$R[g(x)] = \max g(x) \ . \tag{A.2}$$

A functional can be extended to take two or more functions as its arguments just like a normal function can take more than one argument. The use of functionals is ubiquitous in physics e.g. the action in classical mechanics and in the calculus of variations, expectation values in quantum mechanics etc.

As with multi-variable functions where we have extensions to differentiation etc., the concept of a functional generalises many of the results and theorems of multi-variable calculus. Summarised below are three important functional calculus techniques that we use in this thesis.

A.1.1 Functional differentiation

The derivative of a functional $F[g]$ tells us how much $F[g]$ changes for a given small change in g. As with the change of notation ∂ for the infinitesimal in partial differentiation from the d in ordinary differentiation, functional differentiation uses the symbol δ. In most circumstances a functional derivative can be calculated by

$$\frac{\delta F[g(x)]}{\delta g(p)} = \lim_{\epsilon \to 0} \frac{F[g(x) + \epsilon\delta(x-p)] - F[g(x)]}{\epsilon}. \tag{A.3}$$

The extension of this idea to a vector field of vector arguments, as in the case of the NSE velocity field, can easily be shown to be

$$\frac{\delta F[g_\alpha(\mathbf{x},\mathbf{y})]}{\delta g_\beta(\mathbf{p},\mathbf{q})} = \lim_{\epsilon\to 0} \frac{F[g_\alpha(\mathbf{x},\mathbf{y}) + \epsilon\delta_{\alpha\beta}\delta(\mathbf{x}-\mathbf{p})\delta(\mathbf{y}-\mathbf{q})] - F[g_\alpha(\mathbf{x},\mathbf{y})]}{\epsilon} \tag{A.4}$$

Please note that due to the way a functional derivative is defined, the units of a functional derivative are not intuitive. For more details on this and the definitions of functional derivatives see Torralba [136].

A.1.2 Functional Integration

To illustrate the concept of a functional integral it is helpful to go back and look at the case for normal Riemann integrals of functions. In a simple integral such as the one in (A.1), the integral represents a continuum limit version of a sum of all the values taken by $\ln f(x)$ as its argument moves over a line (or set) of points, multiplied by the measure or interval size of the line dx, within a certain interval $[a,b]$. In a functional integral we sum over all the values that a functional takes, as its argument moves over a set of functions, times some measure of the 'volume' of function space associated with each function [135] (now denoted by a big $\mathcal{D}f(x)$ instead of a little dx). For example in the functional integral

$$I = \int \mathcal{D}f(x) A[f(x)] \,, \tag{A.5}$$

where the functional $A[f(x)]$ is given by

$$A[f(x)] = \exp\left\{ -\int_{-L}^{L} dx f^2(x) \right\} \,, \tag{A.6}$$

the functional integral (A.5) can be seen as a sum over all the possible functions that $f(x)$ could be over the interval $[-L,L]$. To understand what this functional integral represents we appeal to the notion that a normal integral is just the continuum limit of a sum so that we can write (A.6) as

$$A(f_1, f_2, \ldots f_n) = \exp\left\{ -\frac{2L}{n} \sum_{i=1}^{n} f_i^2 \right\} . \tag{A.7}$$

This now allows us to write the functional integral (A.5) in a reduced form as

$$\begin{aligned} I_n &= \int df_1 df_2 \dots df_n \exp\left\{-\frac{2L}{n}\sum_{i=1}^{n} f_i^2\right\} \\ &= \prod_{i=1}^{n} \int df_i \exp\left\{-\frac{2L}{n} f_i^2\right\} , \end{aligned} \tag{A.8}$$

where we have used the property of exponent sums to be turned into products outside of the exponent. Again we see that we are effectively summing over all functions that f can take. In some particular cases in physics this sum over functions is seen as a sum over paths and the functional integral is known as a path integral. Please note that the concept of a functional integral has to be handled with care as the limit $\lim_{n\to\infty}$ in the case of the discretised equation (A.7) does not generally exist and in most cases we need some kind of normalising factor to sort out any divergences that occur (see Binney *et. al.* [135]).

A.1.3 Functional Taylor series

Just like we can expand a function $f(x)$ around an arbitrary point x_0, we can also generalise to expanding a functional $G[x(s)]$ around an arbitrary function $y(s)$. This can be seen by noting that an infinitesimal change dx in a function $f(x)$ can be written as

$$df(x) = \frac{df(x)}{d(x)} dx \ . \tag{A.9}$$

For changes which are not infinitesimal we will need higher derivatives to describe the change. In a similar way, for a multi-variable function and change in the function for a given infinitesimal change in the arguments is given by the total differential

$$df(x_1, x_2 \dots x_n) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \ . \tag{A.10}$$

This latter result can be generalised to a functional by taking the continuum limit

$$\delta f[x(t)] = \int dt' \frac{\delta f[x(s)]}{\delta x(t')} \delta x(t') \ . \tag{A.11}$$

Now for non-infinitesimal changes, generalising suitably allows us to write the Taylor series for our original functional $G[x(s)]$ around an arbitrary function $y(s)$

around an interval $[a, b]$

$$\begin{aligned} G\left[x(s)\right] &= \chi_0(s) + \int_a^b \chi_1(t_1)\left(x(t_1) - y(t_1)\right) dt_1 + \\ &+ \int_a^b \int_a^b \chi_2(t_1, t_2)\left(x(t_1) - y(t_1)\right)\left(x(t_2) - y(t_2)\right) dt_1 dt_2 + \\ &+ \cdots + \\ &+ \left[\int_a^b \cdots \int_a^b dt_1 \cdots dt_n \chi_n(t_1, \ldots, t_n) \times \right. \\ &\left. \times \left(x(t_1) - y(t_1)\right) \cdots \left(x(t_n) - y(t_n)\right)\right] , \end{aligned} \tag{A.12}$$

in the limit $n \to \infty$, where the χ's are the susceptibilities given by

$$\chi_n(t_1, \ldots, t_n) = \frac{1}{n!} \left. \frac{\delta^{(n)} G[x(s)]}{\delta x(t_1)\delta x(t_2) \cdots \delta x(t_n)} \right|_{x=y} . \tag{A.13}$$

A.2 Generalised functions

A.2.1 Continuous linear functionals and distributions

A generalised function is an extension of the concept of a function. They are a particular form of functionals known as continuous linear functionals. These are generally defined by noting that if for example we have a functional $F[x(t)]$ say, then if

$$x(t) = \lambda x_1(t) + \mu x_2(t) , \tag{A.14}$$

$F[x(t)]$ is a linear continuous functional if

$$F[x(t)] = \lambda F[x_1(t)] + \mu F[x_2(t)] . \tag{A.15}$$

A generalised function is a linear continuous functional, but not all linear continuous functionals are generalised functions. A generalised function $f(t)$ is usually (but not always) defined by the effect that it has on a test function $\phi(t)$ inside an integral

$$F[\phi(t)] = \int_b^a dt f(t)\phi(t) , \tag{A.16}$$

where one can see that it is defined using continuous linear functionals. Generalised functions are also known as distributions; for example a probability

distribution is a generalised function. The most widely used generalised function is the famous Dirac delta function which is only defined by its action on a test function in an integral.

A.2.2 Causal functions

The Heaviside unit-step function (in literature) is used whenever one encounters causal functions, as in the present work with the causal nature of the propagator in Chapter 8. It is a generalised function and is usually defined in different ways depending on the use. The differences in definition are due to the value it takes when its argument is zero, at the discontinuity. In the present work we are using the definition

$$\begin{aligned} \forall\ \tau \geq 0\ \ \theta\left(\tau\right) &= 1 \\ \forall\ \tau < 0\ \ \theta\left(\tau\right) &= 0. \end{aligned} \tag{A.17}$$

Some properties of the Heaviside function are

$$\theta(t-s)\theta(s-t') = \theta(t-t'), \tag{A.18}$$

$$\frac{\partial}{\partial t}\theta(t-t') = \delta(t-t'), \tag{A.19}$$

$$\theta(t-t')\theta(t'-t) = \delta_{t,t'}, \tag{A.20}$$

where the kronecker delta is defined as

$$\delta_{t,t'} = \begin{cases} 1\ \ \forall t = t' \\ 0\ \ \forall t \neq t'. \end{cases} \tag{A.21}$$

Also

$$\theta(t-t')^n = \theta(t-t'), \tag{A.22}$$

so that

$$\begin{aligned}\theta(t-t')\delta_{t,t'} &= \theta(t-t')\theta(t-t')\theta(t'-t) \\ &= \theta(t-t')^2\theta(t'-t) \\ &= \theta(t-t')\theta(t'-t) \\ &= \delta_{t,t'} \end{aligned} \tag{A.23}$$

Please note that it is important to know that (A.22) is a one way equality inside differentials, so that

$$\frac{\partial}{\partial t}\theta(t-t')^n \rightarrow \frac{\partial}{\partial t}\theta(t-t'), \tag{A.24}$$

but

$$\frac{\partial}{\partial t}\theta(t-t')^n \nleftarrow \frac{\partial}{\partial t}\theta(t-t'). \tag{A.25}$$

If the latter were true it would produce nonsense. As with all relations involving generalised functions, (A.24) and (A.25) can be shown by looking at their actions on a test function inside an integral.

Appendix B

Turbulent energy transfer

This appendix presents the detailed proof of some of the identities that are used in several places in the thesis. The first proof is regarding the conservation of energy in a statistically steady state. More importantly it shows that the non-linear terms in the NSE (the pressure and the quadratic non-linearity) are only responsible for transferring and distributing energy around the system and not into or out of the system. This result, although intuitively expected, is instructive because it is the basis for the other energy flux calculations used in this thesis, especially in Chapter 5. The second proof is for the Detailed Balance identity for velocity mode triads, as first introduced by Kraichnan [67], which is considered a microstate property as it does not involve any averaging over realisations.

B.1 Proof of $\int_0^\infty T(k,t)dk = 0$

We start by noting that we will be using the following shortened notation for averaged velocity mode triad interactions

$$Q_{\alpha\beta\gamma}(-\mathbf{k},\mathbf{j},\mathbf{k}-\mathbf{j};t) = \langle u_\alpha(-\mathbf{k},t)u_\beta(\mathbf{j},t)u_\gamma(\mathbf{k}-\mathbf{j},t)\rangle\,. \tag{B.1}$$

Appendix B. Turbulent energy transfer

Expanding out the $T(k,t)$ term in the energy flux integral

$$\begin{aligned}\int_0^\infty T(k,t)dk &= \int_0^\infty dk 2\pi k^2 M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j \left[Q_{\alpha\beta\gamma}(-\mathbf{k},\mathbf{j},\mathbf{k}-\mathbf{j};t)\right. \\ &\quad \left. -Q_{\alpha\beta\gamma}(\mathbf{k},-\mathbf{j},-\mathbf{k}+\mathbf{j};t)\right] \\ &= \int d^3k \int d^3j \frac{1}{2} M_{\alpha\beta\gamma}(\mathbf{k}) \times \\ &\quad \times \left[Q_{\alpha\beta\gamma}(-\mathbf{k},\mathbf{j},\mathbf{k}-\mathbf{j};t) - Q_{\alpha\beta\gamma}(\mathbf{k},-\mathbf{j},-\mathbf{k}+\mathbf{j};t)\right], \quad \text{(B.2)}\end{aligned}$$

where in the last line (remembering that we are working in an isotropic system) we have used the property of surface integrals

$$\int d^3k = \int_0^\infty dk \int d\Omega(k) = \int_0^\infty dk 4\pi k^2, \quad \text{(B.3)}$$

and where $d\Omega(k)$ is the usual surface measure in spherical polar coordinates.

Next we reduce the $M_{\alpha\beta\gamma}(\mathbf{k})$ operator, looking at the first term in (B.2) as an example

$$\begin{aligned}&M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j \left\langle u_\alpha\left(-\mathbf{k},t\right) u_\beta(\mathbf{j},t) u_\gamma(\mathbf{k}-\mathbf{j},t)\right\rangle \\ &\quad = \frac{1}{2i}\left[k_\beta P_{\alpha\gamma}(\mathbf{k}) + k_\gamma P_{\alpha\beta}(\mathbf{k})\right] \times \\ &\quad \times \int d^3j \left\langle u_\alpha\left(-\mathbf{k},t\right) u_\beta(\mathbf{j},t) u_\gamma(\mathbf{k}-\mathbf{j},t)\right\rangle. \quad \text{(B.4)}\end{aligned}$$

Focusing on the first of the terms in the square brackets, we rename indices using the dummy nature of β and γ

$$\begin{aligned}&k_\beta P_{\alpha\gamma}(\mathbf{k}) \int d^3j \left\langle u_\alpha\left(-\mathbf{k},t\right) u_\beta(\mathbf{j},t) u_\gamma(\mathbf{k}-\mathbf{j},t)\right\rangle \\ &= k_\gamma P_{\alpha\beta}(\mathbf{k}) \int d^3j \left\langle u_\alpha\left(-\mathbf{k},t\right) u_\gamma(\mathbf{j},t) u_\beta(\mathbf{k}-\mathbf{j},t)\right\rangle, \quad \text{(B.5)}\end{aligned}$$

and using the symmetry property of interchanging wavevectors in a convolution [98]

$$\int d^3j u_\gamma(\mathbf{j},t) u_\beta(\mathbf{k}-\mathbf{j},t) = \int d^3j u_\gamma(\mathbf{k}-\mathbf{j},t) u_\beta(\mathbf{j},t). \quad \text{(B.6)}$$

we write (B.5) as

$$\begin{aligned} & k_\gamma P_{\alpha\beta}(\mathbf{k}) \int d^3j \, \langle u_\alpha \left(-\mathbf{k}, t\right) u_\gamma(\mathbf{j}, t) u_\beta(\mathbf{k}-\mathbf{j}, t)\rangle \\ & = k_\gamma P_{\alpha\beta}(\mathbf{k}) \int d^3j \, \langle u_\alpha \left(-\mathbf{k}, t\right) u_\gamma(\mathbf{k}-\mathbf{j}, t) u_\beta(\mathbf{j}, t)\rangle \\ & = k_\gamma P_{\alpha\beta}(\mathbf{k}) \int d^3j \, \langle u_\alpha \left(-\mathbf{k}, t\right) u_\beta(\mathbf{j}, t) u_\gamma(\mathbf{k}-\mathbf{j}, t)\rangle \, . \end{aligned} \tag{B.7}$$

This slight diversion now allows us to write equation (B.4) in the form

$$\begin{aligned} & M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j \, \langle u_\alpha \left(-\mathbf{k}, t\right) u_\beta(\mathbf{j}, t) u_\gamma(\mathbf{k}-\mathbf{j}, t)\rangle \\ & = -ik_\gamma P_{\alpha\beta}(\mathbf{k}) \int d^3j \, \langle u_\alpha \left(-\mathbf{k}, t\right) u_\beta(\mathbf{j}, t) u_\gamma(\mathbf{k}-\mathbf{j}, t)\rangle \, . \end{aligned} \tag{B.8}$$

Making appropriate substitutions in (B.2) using (B.8), we obtain

$$\begin{aligned} \int_0^\infty dk T(k,t) \;\; = \;\; & \frac{1}{2i} \int d^3k \int d^3j k_\gamma P_{\alpha\beta}(\mathbf{k}) \left[Q_{\alpha\beta\gamma}(-\mathbf{k}, \mathbf{j}, \mathbf{k}-\mathbf{j}; t) \right. \\ & \left. - Q_{\alpha\beta\gamma}(\mathbf{k}, -\mathbf{j}, -\mathbf{k}+\mathbf{j}; t) \right] , \end{aligned} \tag{B.9}$$

We then re-write the first term in the integrand of (B.9) as

$$k_\gamma P_{\alpha\beta}(\mathbf{k}) Q_{\alpha\beta\gamma} \left(-\mathbf{k}, \mathbf{j}, \mathbf{k}-\mathbf{j}; t\right) = \langle P_{\alpha\beta}(\mathbf{k}) u_\alpha \left(-\mathbf{k}, t\right) u_\beta(\mathbf{j}, t) k_\gamma u_\gamma(\mathbf{k}-\mathbf{j}, t)\rangle \, . \tag{B.10}$$

Using $P_{\alpha\beta}(\mathbf{k}) u_\alpha \left(-\mathbf{k}, t\right) = u_\beta \left(-\mathbf{k}, t\right)$, and the continuity condition to introduce

$$(k_\gamma - j_\gamma) u_\gamma(\mathbf{k}-\mathbf{j}, t) = 0, \tag{B.11}$$

we can write the above expression as

$$\begin{aligned} & k_\gamma P_{\alpha\beta}(\mathbf{k}) Q_{\alpha\beta\gamma} \left(-\mathbf{k}, \mathbf{j}, \mathbf{k}-\mathbf{j}; t\right) \\ & \quad = \langle u_\beta \left(-\mathbf{k}, t\right) u_\beta(\mathbf{j}, t) \left(k_\gamma - (k_\gamma - j_\gamma)\right) u_\gamma(\mathbf{k}-\mathbf{j}, t)\rangle \\ & \quad = \langle u_\beta \left(-\mathbf{k}, t\right) u_\beta(\mathbf{j}, t) j_\gamma u_\gamma(\mathbf{k}-\mathbf{j}, t)\rangle \\ & \quad = j_\gamma Q_{\beta\beta\gamma} \left(-\mathbf{k}, \mathbf{j}, \mathbf{k}-\mathbf{j}; t\right) . \end{aligned} \tag{B.12}$$

A similar procedure with the second term in equation (B.9) but without the use

of (B.11), allows us to write (B.9) as

$$\int_0^\infty dkT(k,t) = \frac{1}{2i}\int d^3k \int d^3j \, [j_\gamma Q_{\beta\beta\gamma}(-\mathbf{k},\mathbf{j},\mathbf{k}-\mathbf{j};t) - k_\gamma Q_{\beta\beta\gamma}(\mathbf{k},-\mathbf{j},-\mathbf{k}+\mathbf{j};t)] \, . \tag{B.13}$$

On the interchange of $\mathbf{j} \leftrightarrow \mathbf{k}$ dummy wavevector variables on the RHS, we find that equation (B.13) becomes

$$\int_0^\infty dkT(k,t) = \frac{1}{2i}\int d^3k \int d^3j \, [k_\gamma Q_{\beta\beta\gamma}(-\mathbf{j},\mathbf{k},\mathbf{j}-\mathbf{k};t) - j_\gamma Q_{\beta\beta\gamma}(\mathbf{j},-\mathbf{k},-\mathbf{j}+\mathbf{k};t)] \, . \tag{B.14}$$

Re-arranging the velocity modes inside the Q terms we find that

$$\begin{aligned} \int_0^\infty dkT(k,t) &= -\frac{1}{2i}\int d^3k \int d^3j \, [j_\gamma Q_{\beta\beta\gamma}(-\mathbf{k},\mathbf{j},\mathbf{k}-\mathbf{j};t) \\ &\quad - k_\gamma Q_{\beta\beta\gamma}(\mathbf{k},-\mathbf{j},-\mathbf{k}+\mathbf{j};t)] \\ &= -\int_0^\infty dkT(k,t) \, , \end{aligned} \tag{B.15}$$

where we used equation (B.13) in the second line. The result in equation (B.15) shows that the integral is anti-symmetric under interchange of dummy variables and thus vanishes and proves the desired result that

$$\int d^3kT(k,t) = 0. \tag{B.16}$$

This illustrates that the non-linear transfer term is conservative i.e. its role is to transfer energy amongst the Fourier modes only. To conclude this section, it is of interest to note that this proof essentially requires that the integral limits on the k and j integrals be the same, so the dummy wavevector variables can be interchanged.

B.2 Proof of the Detailed Balance identity

The Detailed Balance identity relates the sum of the velocity mode triad interactions with their associate wavenumbers permuted

$$\tilde{T}(-k|j,l) + \tilde{T}(l|-k,j) + \tilde{T}(j|l,-k) = 0, \tag{B.17}$$

where $l = |\mathbf{k} - \mathbf{j}|$ is needed to satisfy the wavenumber triad (from homogeneity), and where

$$\tilde{T}(-k|j,l) = M_{\alpha\beta\gamma}(\mathbf{k}) \left[Q_{\alpha\beta\gamma}(-\mathbf{k},\mathbf{j},\mathbf{l};t) - Q_{\alpha\beta\gamma}(\mathbf{k},-\mathbf{j},-\mathbf{l};t) \right]. \tag{B.18}$$

The proof of this identity can be seen by expanding the RHS of (B.17) using (B.18)

$$\begin{aligned}
&\tilde{T}(-k|j,l) + \tilde{T}(l|-k,j) + \tilde{T}(j|l,-k) \\
&= M_{\alpha\beta\gamma}(\mathbf{k}) \left[Q_{\alpha\beta\gamma}(-\mathbf{k},\mathbf{j},\mathbf{l};t) - Q_{\alpha\beta\gamma}(\mathbf{k},-\mathbf{j},-\mathbf{l};t) \right] + \\
&+ M_{\alpha\beta\gamma}(-\mathbf{l}) \left[Q_{\alpha\beta\gamma}(\mathbf{l},-\mathbf{k},\mathbf{j};t) - Q_{\alpha\beta\gamma}(-\mathbf{l},\mathbf{k},-\mathbf{j};t) \right] + \\
&+ M_{\alpha\beta\gamma}(-\mathbf{j}) \left[Q_{\alpha\beta\gamma}(\mathbf{j},\mathbf{l},-\mathbf{k};t) - Q_{\alpha\beta\gamma}(-\mathbf{j},-\mathbf{l},\mathbf{k};t) \right].
\end{aligned} \tag{B.19}$$

Remembering that $M_{\alpha\beta\gamma}(-\mathbf{k}) = -M_{\alpha\beta\gamma}(\mathbf{k})$ we rearrange (B.19) and group into two classes of terms

$$\begin{aligned}
&\tilde{T}(-k|j,l) + \tilde{T}(l|-k,j) + \tilde{T}(j|l,-k) \\
&= \{ M_{\alpha\beta\gamma}(\mathbf{k}) Q_{\alpha\beta\gamma}(-\mathbf{k},\mathbf{j},\mathbf{l};t) - M_{\alpha\beta\gamma}(\mathbf{l}) Q_{\alpha\beta\gamma}(\mathbf{l},-\mathbf{k},\mathbf{j};t) \\
&- M_{\alpha\beta\gamma}(\mathbf{j}) Q_{\alpha\beta\gamma}(\mathbf{j},\mathbf{l},-\mathbf{k};t) \}_A + \{ -M_{\alpha\beta\gamma}(\mathbf{k}) Q_{\alpha\beta\gamma}(\mathbf{k},-\mathbf{j},-\mathbf{l};t) \\
&+ M_{\alpha\beta\gamma}(\mathbf{l}) Q_{\alpha\beta\gamma}(-\mathbf{l},\mathbf{k},-\mathbf{j};t) + M_{\alpha\beta\gamma}(\mathbf{j}) Q_{\alpha\beta\gamma}(-\mathbf{j},-\mathbf{l},\mathbf{k};t) \}_B .
\end{aligned} \tag{B.20}$$

Remembering that the Q term involves velocity Fourier modes which are complex, we can see that the terms in brace B are the complex conjugates of the terms in brace A. We will prove the Detailed Balance identity for A; the proof for B follows in the same way i.e. they are both equal to zero separately.

To proceed we expand the M operators and write each term in A on the RHS

of equation (B.20) in terms of the velocity modes

$$\begin{aligned}\{\}_A &= \frac{1}{2i}\left[k_\beta P_{\alpha\gamma}(\mathbf{k}) + k_\gamma P_{\alpha\beta}(\mathbf{k})\right]\left\langle u_\alpha\left(-\mathbf{k},t\right)u_\beta(\mathbf{j},t)u_\gamma(\mathbf{l},t)\right\rangle \\ &\quad -\frac{1}{2i}\left[l_\beta P_{\alpha\gamma}(\mathbf{l}) + l_\gamma P_{\alpha\beta}(\mathbf{l})\right]\left\langle u_\alpha\left(-\mathbf{l},t\right)u_\beta(-\mathbf{k},t)u_\gamma(\mathbf{j},t)\right\rangle \\ &\quad -\frac{1}{2i}\left[j_\beta P_{\alpha\gamma}(\mathbf{j}) + j_\gamma P_{\alpha\beta}(\mathbf{j})\right]\left\langle u_\alpha\left(-\mathbf{j},t\right)u_\beta(-\mathbf{k},t)u_\gamma(\mathbf{l},t)\right\rangle \\ &= \frac{1}{2i}\left\{k_\beta\left\langle u_\gamma\left(-\mathbf{k},t\right)u_\beta(\mathbf{j},t)u_\gamma(\mathbf{l},t)\right\rangle + k_\gamma\left\langle u_\beta\left(-\mathbf{k},t\right)u_\beta(\mathbf{j},t)u_\gamma(\mathbf{l},t)\right\rangle\right. \\ &\quad -l_\beta\left\langle u_\gamma\left(-\mathbf{l},t\right)u_\beta(-\mathbf{k},t)u_\gamma(\mathbf{j},t)\right\rangle - l_\gamma\left\langle u_\beta\left(-\mathbf{l},t\right)u_\beta(-\mathbf{k},t)u_\gamma(\mathbf{j},t)\right\rangle \\ &\quad \left.-j_\beta\left\langle u_\gamma\left(-\mathbf{j},t\right)u_\beta(-\mathbf{k},t)u_\gamma(\mathbf{l},t)\right\rangle - j_\gamma\left\langle u_\beta\left(-\mathbf{j},t\right)u_\beta(-\mathbf{k},t)u_\gamma(\mathbf{l},t)\right\rangle\right\},\end{aligned} \tag{B.21}$$

where in the second equality we used the property $P_{\alpha\beta}(\mathbf{k})u_\alpha(\mathbf{k},t) = u_\beta(\mathbf{k},t)$. Now pairing the first and fourth terms we get

$$\begin{aligned}&k_\beta\left\langle u_\gamma\left(-\mathbf{k},t\right)u_\beta(\mathbf{j},t)u_\gamma(\mathbf{l},t)\right\rangle - l_\gamma\left\langle u_\beta\left(-\mathbf{l},t\right)u_\beta(-\mathbf{k},t)u_\gamma(\mathbf{j},t)\right\rangle \\ &= k_\beta\left\langle u_\gamma\left(-\mathbf{k},t\right)u_\beta(\mathbf{j},t)u_\gamma(\mathbf{l},t)\right\rangle - l_\beta\left\langle u_\gamma\left(-\mathbf{l},t\right)u_\gamma(-\mathbf{k},t)u_\beta(\mathbf{j},t)\right\rangle,\end{aligned} \tag{B.22}$$

where in the second line we have renamed the indices of the second term as they are dummies. Re-arranging the velocity modes in the second term, we see that we can write (B.22) as

$$\begin{aligned}&(k_\beta - l_\beta)\left\langle u_\gamma\left(-\mathbf{k},t\right)u_\beta(\mathbf{j},t)u_\gamma(\mathbf{l},t)\right\rangle \\ &\quad = (k_\beta - (k_\beta - j_\beta))\left\langle u_\gamma\left(-\mathbf{k},t\right)u_\beta(\mathbf{j},t)u_\gamma(\mathbf{l},t)\right\rangle \\ &\quad = \left\langle u_\gamma\left(-\mathbf{k},t\right)j_\beta u_\beta(\mathbf{j},t)u_\gamma(\mathbf{l},t)\right\rangle \\ &\quad = 0\,,\end{aligned} \tag{B.23}$$

where the last equality follows from the continuity condition.

In the same way, the second and fifth terms, and the third and sixth terms in equation (B.21) can be paired and shown to be equal to zero. As said before, we can go through the same procedure with the terms contained within brace B in equation (B.20) and this will also result in zero, which proves the desired result in the Detailed Balance identity (B.17). It is easily seen that the Detailed Balance identity can be proved independently from the average (represented by the angle brackets) i.e. it holds for each realisation of the turbulent ensemble; it is a microstate property.

Appendix B. Turbulent energy transfer

The Detailed Balance identity can also be used to prove (B.16) by noting that if we construct the spectral energy balance equation for $E(k,t)$, $E(j,t)$ and $E(l,t)$, and then integrate them over k, j and l respectively to obtain the flux, then adding these equations together with a suitable re-labeling of dummy variables allows us to write

$$\begin{aligned}\frac{\partial E(t)}{\partial t} &+ \int_0^\infty dk 2\nu k^2 E(k,t) \\ &= \frac{1}{3}\int_0^\infty dk \int_0^\infty dj \int_0^\infty dl \delta(k-j-l) \times \\ &\quad \times \left[\tilde{T}(-k|j,l) + \tilde{T}(l|-k,j) + \tilde{T}(j|l,-k)\right] \\ &\equiv \int_0^\infty dk T(k,t) = 0 \ , \end{aligned} \tag{B.24}$$

where the last result just follows from using the Detailed Balance identity.

Appendix C

Detailed proofs of LET results

C.1 Velocity based formulation of LET propagator

- From the exact solution of the solenoidal NSE

$$\begin{aligned} u_{\alpha}\left(\mathbf{k},t\right) &= \widehat{H}_{\alpha\sigma}^{(0)}\left(\mathbf{k};t,s\right)u_{\sigma}\left(\mathbf{k},s\right)+ \\ &\quad +\lambda\int_{s}^{t}dt''\widehat{H}_{\alpha\sigma}^{(0)}\left(\mathbf{k};t,t''\right)\int d^{3}jM_{\sigma\beta\gamma}(\mathbf{k})u_{\beta}\left(\mathbf{j},t''\right)u_{\gamma}\left(\mathbf{k}-\mathbf{j},t''\right). \end{aligned} \tag{C.1}$$

- Expanding $u_{\alpha}\left(\mathbf{k},t\right)$ in a perturbation series and equating zero-order terms we can say that the equality

$$u_{\alpha}^{(0)}\left(\mathbf{k},t\right)=\widehat{H}_{\alpha\sigma}^{(0)}\left(\mathbf{k};t,s\right)u_{\alpha}^{(0)}\left(\mathbf{k},s\right), \tag{C.2}$$

  illustrates the propagator like nature of $\widehat{H}_{\alpha\sigma}^{(0)}\left(\mathbf{k};t,s\right)$.

- From looking at the form of (C.1), we then ***postulate*** the existence of a **renormalized propagator** such that we obtain a renormalized version of (C.2)

$$u_{\alpha}\left(\mathbf{k},t\right)=\widehat{H}_{\alpha\sigma}\left(\mathbf{k};t,s\right)u_{\alpha}\left(\mathbf{k},s\right). \tag{C.3}$$

- Multiply (C.3) by $u_{\beta}\left(-\mathbf{k},t'\right)$

$$u_{\alpha}\left(\mathbf{k},t\right)u_{\beta}\left(-\mathbf{k},t'\right)=\widehat{H}_{\alpha\sigma}\left(\mathbf{k};t,s\right)u_{\sigma}\left(\mathbf{k},s\right)u_{\beta}\left(-\mathbf{k},t'\right) \tag{C.4}$$

Average this equation to get

$$Q_{\alpha\beta}(\mathbf{k};t,t') = H_{\alpha\sigma}\left(\mathbf{k};t,s\right) Q_{\sigma\beta}\left(\mathbf{k};s,t'\right) \tag{C.5}$$

where the propagator is statistically independent of the velocity field and we have used the mean-field approximation

$$\left\langle \widehat{H}_{\alpha\sigma}\left(\mathbf{k};t,s\right)\right\rangle = H_{\alpha\sigma}\left(\mathbf{k};t,s\right). \tag{C.6}$$

- (C.5) can be turned into a simpler scalar form by using the property of isotropic tensors given in equation (1.19)

$$Q\left(k;t,t'\right) = \theta\left(t-s\right) H\left(k;t,s\right) Q\left(k;s,t'\right). \tag{C.7}$$

C.1.1 Transitivity of the velocity field propagator

- This can be proven by applying (C.3) on the RHS of itself

$$u_{\alpha}\left(\mathbf{k},t\right) = \widehat{H}_{\alpha\sigma}\left(\mathbf{k};t,s\right) \widehat{H}_{\sigma\rho}\left(\mathbf{k};s,p\right) u_{\rho}\left(\mathbf{k},p\right), \tag{C.8}$$

and realising that we could also have written this as

$$u_{\alpha}\left(\mathbf{k},t\right) = \widehat{H}_{\alpha\rho}\left(\mathbf{k};t,p\right) u_{\rho}\left(\mathbf{k},p\right), \tag{C.9}$$

implying the desired result

$$\widehat{H}_{\alpha\rho}\left(\mathbf{k};t,p\right) = \widehat{H}_{\alpha\sigma}\left(\mathbf{k};t,s\right) \widehat{H}_{\sigma\rho}\left(\mathbf{k};s,p\right). \tag{C.10}$$

- We can also make a further statement about the nature of the propagator. This is that previously [118, 108] we assumed the mean-field result (C.6) to obtain (C.7). However, since in Chapter 8 we have now done the entire analysis with correlators, the mean-field result is no longer an assumption.

C.1.2 Linking two single-time correlators

- We can expand a single-time correlation using (C.3)

$$\begin{aligned} Q_{\alpha\alpha}(\mathbf{k};t,t) &= \langle u_\alpha(-\mathbf{k},t)u_\alpha(\mathbf{k},t)\rangle \\ &= \left\langle \widehat{H}_{\alpha\sigma}\left(-\mathbf{k};t,s\right)u_\sigma(-\mathbf{k},s)\widehat{H}_{\alpha\rho}\left(\mathbf{k};t,s\right)u_\rho(\mathbf{k},s)\right\rangle, \end{aligned} \tag{C.11}$$

  using (C.6) we can write this as

$$Q_{\alpha\alpha}(\mathbf{k};t,t) = H_{\alpha\sigma}\left(-\mathbf{k};t,s\right)H_{\alpha\rho}\left(\mathbf{k};t,s\right)Q_{\sigma\rho}(\mathbf{k};s,s). \tag{C.12}$$

  Putting this into it's isotropic form we get the desired result

$$Q(k;t,t) = H\left(k;t,s\right)H\left(k;t,s\right)Q(k;s,s). \tag{C.13}$$

C.2 Proof of the isotropic two-time correlator exhibiting invariance under interchange of time arguments

- The Correlation function of two velocities at different times t and t' is given by

$$Q_{\alpha\beta}(\mathbf{k};t,t') = \langle u_\alpha(-\mathbf{k},t)u_\beta(\mathbf{k},t')\rangle \tag{C.14}$$

- Isotropic tensors have the property

$$Q_{\alpha\beta}(\mathbf{k};t,t') = Q_{\beta\text{`}\alpha}(\mathbf{k};t,t') \tag{C.15}$$

  Using this property we can write

$$\begin{aligned} Q_{\alpha\beta}(\mathbf{k};t,t') &= Q_{\beta\text{`}\alpha}(\mathbf{k};t,t') \\ &= \langle u_\beta(-\mathbf{k},t)u_\alpha(\mathbf{k},t')\rangle \\ &= \langle u_\alpha(\mathbf{k},t')u_\beta(-\mathbf{k},t)\rangle \\ &= Q_{\alpha\beta}(-\mathbf{k};t',t), \end{aligned} \tag{C.16}$$

so we are left with

$$Q_{\alpha\beta}(\mathbf{k};t,t') = Q_{\alpha\beta}(-\mathbf{k};t',t). \tag{C.17}$$

- Another property of isotropic tensors is

$$I_{\alpha\beta}(\mathbf{x}) = P_{\alpha\beta}(\mathbf{x})I(x), \tag{C.18}$$

where $I_{\alpha\beta}(\mathbf{x})$ is an arbitrary isotropic tensor. Using this property of isotropic tensors we can write (C.17) as

$$P_{\alpha\beta}(\mathbf{k})Q(k;t,t') = P_{\alpha\beta}(-\mathbf{k})Q(k;t',t), \tag{C.19}$$

where

$$P_{\alpha\beta}(\mathbf{k}) = \delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{|k|^2}. \tag{C.20}$$

Using the fact that $P_{\alpha\beta}(-\mathbf{k}) = P_{\alpha\beta}(\mathbf{k})$ we can finally write the result

$$Q(k;t,t') = Q(k;t',t) \tag{C.21}$$

showing that the two-time correlation function is invariant under interchange of the time arguments.

- An associated result for **stationary** isotropic correlators that uses (C.21) as its proof is that

$$Q(k;t-t') = Q(k;t'-t), \tag{C.22}$$

and if we say that $\tau = t - t'$ we can write

$$Q(k;\tau) = Q(k;-\tau), \tag{C.23}$$

showing that the **stationary isotropic correlator is time-reversal symmetric**.

C.3 Proof of $lim_{\tau\to 0}\frac{\partial Q(\tau)}{\partial \tau} = 0$ for the stationary isotropic correlator in an exponential representation

- We will start by using (8.24) in (8.22) to get

$$\begin{aligned} Q\left(k;t,t'\right) &= \theta\left(t-t'\right) exp\left\{-\omega(k)(t-t')\right\} Q\left(k;t',t'\right) \\ &\quad +\theta\left(t'-t\right) exp\left\{-\omega(k)(t'-t)\right\} Q\left(k;t,t\right) \\ &\quad -\delta_{t,t'} Q\left(k;t,t'\right). \end{aligned} \tag{C.24}$$

- Using the property of stationary turbulence

$$Q\left(k;t,t\right) = Q(k), \tag{C.25}$$

and the representation for $\delta_{t,t'}$

$$\delta_{t,t'} = \theta\left(t-t'\right)\theta\left(t'-t\right) \tag{C.26}$$

(C.24) becomes

$$\begin{aligned} Q\left(k;t,t'\right) &= \underbrace{\theta\left(t-t'\right) exp\left\{-\omega(k)(t-t')\right\} Q\left(k\right)}_{a} \\ &\quad + \underbrace{\theta\left(t'-t\right) exp\left\{-\omega(k)(t'-t)\right\} Q\left(k\right)}_{b} \\ &\quad - \underbrace{\theta\left(t-t'\right)\theta\left(t'-t\right) Q\left(k;t,t'\right)}_{c}. \end{aligned} \tag{C.27}$$

- We can now proceed by differentiating wrt t piece-wise

$$\begin{aligned} \frac{\partial}{\partial t}(a) &= \theta\left(t-t'\right) Q\left(k\right) \frac{\partial}{\partial t} exp\left\{-\omega(k)(t-t')\right\} \\ &\quad + exp\left\{-\omega(k)(t-t')\right\} Q\left(k\right) \frac{\partial}{\partial t}\theta\left(t-t'\right), \end{aligned} \tag{C.28}$$

where we have used the product rule to separate terms. Now we do the

differentiation

$$\begin{aligned}\frac{\partial}{\partial t}(a) &= -\theta\left(t-t'\right)Q\left(k\right)\omega(k)exp\left\{-\omega(k)(t-t')\right\} \\ &\quad +exp\left\{-\omega(k)(t-t')\right\}Q\left(k\right)\delta\left(t-t'\right). \end{aligned} \tag{C.29}$$

- In a similar way

$$\begin{aligned}\frac{\partial}{\partial t}(b) &= \theta\left(t'-t\right)Q\left(k\right)\omega(k)exp\left\{-\omega(k)(t'-t)\right\} \\ &\quad -exp\left\{-\omega(k)(t'-t)\right\}Q\left(k\right)\delta\left(t'-t\right). \end{aligned} \tag{C.30}$$

- For the last differentiation we have

$$\begin{aligned}\frac{\partial}{\partial t}(c) &= \theta\left(t-t'\right)\frac{\partial}{\partial t}\theta\left(t'-t\right)Q\left(k;t,t'\right) \\ &\quad +\theta\left(t'-t\right)\frac{\partial}{\partial t}\theta\left(t-t'\right)Q\left(k;t,t'\right), \end{aligned} \tag{C.31}$$

where we have again used the product rule. We now need to use (8.16) to get

$$\begin{aligned}\frac{\partial}{\partial t}(c) &= \theta\left(t-t'\right)\frac{\partial}{\partial t}\theta\left(t'-t\right)H\left(k;t',t\right)Q\left(k;t,t\right) \\ &+ \theta\left(t'-t\right)\frac{\partial}{\partial t}\theta\left(t-t'\right)H\left(k;t,t'\right)Q\left(k;t',t'\right) \\ &= \theta\left(t-t'\right)\frac{\partial}{\partial t}\theta\left(t'-t\right)exp\left\{-\omega(k)(t'-t)\right\}Q\left(k\right) \\ &+ \theta\left(t'-t\right)\frac{\partial}{\partial t}\theta\left(t-t'\right)exp\left\{-\omega(k)(t-t')\right\}Q\left(k\right), \end{aligned} \tag{C.32}$$

where in the last step we used (8.24). After extensive use of the product rule and then differentiating we obtain

$$\begin{aligned}\frac{\partial}{\partial t}(c) &= \delta_{t,t'}Q\left(k\right)\omega(k)exp\left\{-\omega(k)(t'-t)\right\} \\ &- \theta(t-t')Q(k)exp\left\{-\omega(k)(t'-t)\right\}\delta\left(t'-t\right) \\ &- \delta_{t,t'}Q\left(k\right)\omega(k)exp\left\{-\omega(k)(t-t')\right\} \\ &+ \theta\left(t'-t\right)Q\left(k\right)exp\left\{-\omega(k)(t-t')\right\}\delta\left(t-t'\right). \end{aligned} \tag{C.33}$$

- If we now collate all the terms and take the limit $t \to t'$ we get

$$\begin{aligned}\lim_{t\to t'} \frac{\partial}{\partial t} Q\left(k;t,t'\right) &= -Q\left(k\right)\omega(k) + Q\left(k\right)\lim_{t\to t'}\delta\left(t-t'\right) \\ &\quad +Q\left(k\right)\omega(k) - Q\left(k\right)\lim_{t\to t'}\delta\left(t'-t\right) \\ &\quad +Q\left(k\right)\omega(k) - Q(k)\lim_{t\to t'}\delta\left(t'-t\right) \\ &\quad -Q\left(k\right)\omega(k) + Q\left(k\right)\lim_{t\to t'}\delta\left(t-t'\right). \end{aligned} \tag{C.34}$$

  Using the fact that the delta function behaves like an even function, namely $\delta(t-t') = \delta(t'-t)$, we can see that all the terms cancel out and we are left with the desired result

$$\lim_{t\to t'} \frac{\partial}{\partial t} Q\left(k;t,t'\right) = 0. \tag{C.35}$$

Appendix D

Published papers

On the following pages we reproduce the paper:

Time-ordered fluctuation-dissipation relation for incompressible isotropic turbulence, K. Kiyani and W. D. McComb, Phys. Rev. E **70**, 066303 (2004).

Permission to reproduce the paper on the following pages was granted by the APS. The paper may be downloaded from the Physical Review E homepage at `http://www.pre.aps.org`

Appendix D. Published papers

PHYSICAL REVIEW E **70**, 066303 (2004)

Time-ordered fluctuation-dissipation relation for incompressible isotropic turbulence

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The Kraichnan-Wyld perturbation expansion is used to justify the introduction of a renormalized response function connecting two-point covariances at different times. The resulting relationship was specialized by a suitable choice of initial conditions to the form of a fluctuation-dissipation relation (FDR). This was further developed to reconcile the time symmetry of the covariance with the causality of the response by the introduction of time ordering along with a counterterm. This formulation provides a solution to an old problem, that of representing the time dependence of the covariance and response by exponential forms. We show that the derivative (with respect to difference time) of the covariance now vanishes at the origin. This allows one to study the relationships between two-time spectral closures and time-independent theories such as the self-consistent field theory of Edwards or the more recent renormalization group approaches. We also show that the renormalized response function is transitive with respect to intermediate times and report a different Langevin equation model for turbulence. We note the potential value of this time-ordering procedure in all applications of the FDR.

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Most renormalized spectral theories of turbulence have been based on the Kraichnan-Wyld perturbation theory [1,2]. Let us denote the two-time covariance of the fluctuating turbulent velocity field for mode k by $C(k;t,t')$, where the dependence is only on the scalar wave number due to the assumption of isotropy. The covariance is related to the usual energy spectrum $E(k,t)$ by

$$E(k,t) = 4\pi k^2 C(k;t,t). \tag{1}$$

Isotropy also implies time-reversal symmetry, which requires that

$$C(k;t,t') = C(k;t',t). \tag{2}$$

Reversion of the primitive perturbation series, obtained by iterating the Navier-Stokes equation (NSE) for the bare covariance $C^{(0)}(k;t,t')$ (which has a multivariate normal distribution and is not an observable) in terms of the viscous response (which is an observable), leads to coupled expansions for the exact covariance $C(k;t,t')$ and a renormalized response function $R(k;t,t')$ (say). The renormalized response is not an observable but must nevertheless satisfy the causality condition

$$R(k;t,t') = 0 \text{ for } t' > t. \tag{3}$$

Specific theories are obtained by introducing a specific choice of $R(k;t,t')$ and truncating the renormalized expansions at some low order. The first such theory was the Eulerian [20] direct interaction approximation (DIA: Ref. [1]), in which the response to small perturbations in the forcing (noise) is renormalized. Other pioneering theories are the self-consistent field theories [3,4]. These theories are time independent [21] and the renormalized response is expressed in terms of the eddy decay rate $\omega(k)$. It was shown [5] (see also Ref. [6] for a discussion) that a connection could be made between these approaches by considering the steady state [where $C(k;t,t')=C(k;t-t')$], and assuming exponential forms for the covariance and renormalized response function, thus

$$C(k;t-t') = C(k)e^{-\omega(k)|t-t'|}; \tag{4}$$

$$R(k;t-t') = e^{-\omega(k)(t-t')}. \tag{5}$$

However, there is a basic problem with these forms in that the time-reversal symmetry of Eq. (2) is in practice not satisfied, and that differentiating the steady-state covariance with respect to difference time leads to a nonzero result at the origin, where $t=t'$ (see Ref. [6]). In this paper we will show that a consideration of time ordering in the renormalized response can allow the use of exponential time dependences without encountering these problems.

We begin by considering the generalized covariance equation, as derived from the NSE [7], thus

$$\left[\frac{\partial}{\partial t} + \nu k^2\right] C_{\alpha\sigma}(\mathbf{k};t,t') = \lambda M_{\alpha\beta\gamma}(\mathbf{k}) \times \int d^3j \langle u_\beta(\mathbf{j},t) u_\gamma(\mathbf{k}-\mathbf{j},t) u_\sigma(-\mathbf{k},t')\rangle. \tag{6}$$

The Greek indices are just the usual Cartesian tensor indices relating to the space dimensions and take the values 1, 2, or 3. The inertial transfer operator $M_{\alpha\beta\gamma}(\mathbf{k})$ (see, for example, Ref. [7]) is given by

$$M_{\alpha\beta\gamma}(\mathbf{k}) = \frac{1}{2i}[k_\beta P_{\alpha\gamma} + k_\gamma P_{\alpha\beta}], \tag{7}$$

where the projector $P_{\alpha\beta}(\mathbf{k})$ is given by

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$$P_{\alpha\beta}(\mathbf{k}) = \delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}. \tag{8}$$

By using an integrating factor and integrating over time we can write this as

$$C_{\alpha\sigma}(\mathbf{k};t,t') = R^{(0)}_{\alpha\varepsilon}(\mathbf{k};t,s)C_{\epsilon\sigma}(\mathbf{k};s,t') + \left[\lambda\int_s^t dt'' R^{(0)}_{\alpha\epsilon}(\mathbf{k};t,t'')M_{\epsilon\beta\gamma}(\mathbf{k}) \times \int d^3j\langle u_\beta(\mathbf{j},t'')u_\gamma(\mathbf{k}-\mathbf{j},t'')u_\sigma(-\mathbf{k},t')\rangle\right], \tag{9}$$

where s is some initial time and the integrating factor is

$$R^{(0)}_{\alpha\epsilon}(\mathbf{k};t,t'') = \begin{cases} P_{\alpha\epsilon}(\mathbf{k})e^{-\nu k^2(t-t'')} & t \geqslant t'', \\ 0 & t < t''. \end{cases} \tag{10}$$

From the primitive perturbation series [1,2], we have

$$C_{\alpha\sigma}(\mathbf{k};t,t') = C^{(0)}_{\alpha\sigma}(\mathbf{k};t,t') + \lambda^2 C^{(2)}_{\alpha\sigma}(\mathbf{k};t,t')\cdots. \tag{11}$$

When Eq. (11) is substituted in Eq. (9) we can see that $R^{(0)}_{\alpha\epsilon}(\mathbf{k};t,s)$ acts as a zero-order response for the zero-order covariance, thus

$$C^{(0)}_{\alpha\sigma}(\mathbf{k};t,t') = \theta(t-s)R^{(0)}_{\alpha\epsilon}(\mathbf{k};t,s)C^{(0)}_{\epsilon\sigma}(\mathbf{k};s,t'). \tag{12}$$

This is an exact result. We shall call this the zero-order or bare result. Re-arranging Eq. (9) to prompt the next step,

$$C_{\alpha\sigma}(\mathbf{k};t,t') = \left[R^{(0)}_{\alpha\varepsilon}(\mathbf{k};t,s) + \frac{1}{C_{\epsilon\sigma}(\mathbf{k};s,t')}\lambda\int_s^t dt'' R^{(0)}_{\alpha\epsilon}(\mathbf{k};t,t'')M_{\epsilon\beta\gamma}(\mathbf{k}) \times \int d^3j\langle u_\beta(\mathbf{j},t'')u_\gamma(\mathbf{k}-\mathbf{j},t'')u_\sigma(-\mathbf{k},t')\rangle\right] \times C_{\epsilon\sigma}(\mathbf{k};s,t'), \tag{13}$$

we postulate that we may write this in its renormalized form as

$$C_{\alpha\sigma}(\mathbf{k};t,t') = \theta(t-s)R_{\alpha\epsilon}(\mathbf{k};t,s)C_{\epsilon\sigma}(\mathbf{k};s,t'), \tag{14}$$

or in its *isotropic* version as

$$C(k;t,t') = \theta(t-s)R(k;t,s)C(k;s,t'), \tag{15}$$

where the $\theta(t-s)$ incorporates the causality condition. We have effectively replaced the zero-order equation (12) by its renormalized version using the replacements

$$C^{(0)} \to C,$$

$$R^{(0)} \to R. \tag{16}$$

As yet we have made no choice about the time ordering of the two times t and t', and thus the symmetry under interchange of t and t' is untested in Eq. (15). If we explicitly choose the time ordering as $t > t'$ say, then this is equivalent to applying the Heaviside unit-step function $\theta(t-t')$ to both sides of Eq. (15)

$$\theta(t-t')C(k;t,t') = \theta(t-t')\theta(t-s)R(k;t,s)C(k;s,t'). \tag{17}$$

If we now set $s=t'$ in Eq. (17), which amounts to a choice of the initial condition, we get

$$\theta(t-t')C(k;t,t') = \theta(t-t')R(k;t,t')C(k;t',t'). \tag{18}$$

This result takes the form of a fluctuation-dissipation relationship (or FDR). Of course such relationships are most familiar in microscopic systems at thermal equilibrium but over the years there has been quite some discussion of the way in which relationships like this occur in turbulence theory (for example, see Ref. [8] and references therein). More recently Frederiksen and Davies [9] have distinguished between spectral theories by the way in which relationships of the form of Eq. (18) play a part.

We now introduce a representation of the covariance which preserves the symmetry under interchange of time arguments as

$$C(k;t,t') = \theta(t-t')C(k;t,t') + \theta(t'-t)C(k;t,t') - \delta_{t,t'}C(k;t,t'). \tag{19}$$

Using Eq. (17) to expand the right-hand side of Eq. (19) we obtain

$$C(k;t,t') = \theta(t-t')\theta(t-s)R(k;t,s)C(k;s,t') + \theta(t'-t)\theta(t'-p)R(k;t',p)C(k;p,t) - \delta_{t,t'}C(k;t,t'). \tag{20}$$

Equation (20) may be written in the form of a time-ordered fluctuation-dissipation relation by using Eq. (18) to construct it instead,

$$C(k;t,t') = \theta(t-t')R(k;t,t')C(k;t',t') + \theta(t'-t)R(k;t',t)C(k;t,t) - \delta_{t,t'}C(k;t,t'). \tag{21}$$

The symmetry of both these covariances, Eqs. (20) and (21), can be broken simply by applying a unit-step function to both sides. This will yield something like Eq. (17) or (18), respectively, depending on which time ordering is chosen.

Turning now to the problem of the exponential forms as given by Eqs. (4) and (5) we find that this time-ordered representation (19) has the property that

$$\lim_{t\to t'}\frac{\partial}{\partial t}C(k;t,t') = 0, \tag{22}$$

as it should for stationary turbulence. We can show this result by using Eq. (5) in Eq. (21) to obtain

$$C(k;t,t') = \theta(t-t')\exp\{-\omega(k)(t-t')\}C(k;t',t') + \theta(t'-t)\exp\{-\omega(k)(t'-t)\}C(k;t,t) - \delta_{t,t'}C(k;t,t'). \tag{23}$$

Using the property of stationary turbulence

$$C(k;t,t) = C(k), \tag{24}$$

and the representation for $\delta_{t,t'}$

$$\delta_{t,t'} = \theta(t-t')\theta(t'-t), \tag{25}$$

Eq. (23) becomes

$$\begin{aligned} C(k;t,t') = {} & \theta(t-t')\exp\{-\omega(k)(t-t')\}C(k) \\ & + \theta(t'-t)\exp\{-\omega(k)(t'-t)\}C(k) \\ & - \theta(t-t')\theta(t'-t)C(k;t,t'). \end{aligned} \tag{26}$$

Taking the derivative with respect to t, substituting Eq. (21) in places, collecting all the terms and taking the limit $t \to t'$, we find

$$\begin{aligned} \lim_{t\to t'} \frac{\partial}{\partial t} C(k;t,t') = {} & -C(k)\omega(k) + C(k) \lim_{t\to t'} \delta(t-t') \\ & + C(k)\omega(k) - C(k) \lim_{t\to t'} \delta(t'-t) \\ & + C(k)\omega(k) - C(k) \lim_{t\to t'} \delta(t'-t) \\ & - C(k)\omega(k) + C(k) \lim_{t\to t'} \delta(t-t'). \end{aligned} \tag{27}$$

Using the fact that the δ function behaves like an even function, namely $\delta(t-t') = \delta(t'-t)$, we can see that the terms cancel in pairs and we are left with the desired result Eq. (22).

In contrast to this, the use of Eq. (4) by Leslie failed to obtain this result [6]. This is because the representation in Eq. (21) exhibits the time-reversal symmetry $t \leftrightarrow t'$ in a more manifest way than Eq. (4). Leslie, for his calculation regarding the time derivative of the stationary covariance at the origin, takes $t > t'$ for the representation of the covariance, whereas we have determined in Eq. (2) that the covariance is symmetric under interchange of t and t'.

Finally, we can show that the renormalized response is transitive with respect to intermediate times. Equating the right-hand side of Eq. (17) with the right-hand side of Eq. (18) we obtain

$$\begin{aligned} \theta(t-t')R(k;t,t')C(k;t',t') = {} & \theta(t-t')\theta(t-s) \\ & \times R(k;t,s)C(k;s,t'). \end{aligned} \tag{28}$$

Expanding the right-hand side of Eq. (28) using Eq. (21) we have

$$\begin{aligned} & \theta(t-t')R(k;t,t')C(k;t',t') \\ & = \{[\theta(t-t')\theta(t-s)R(k;t,s) \\ & \quad \times \theta(s-t')R(k;s,t')C(k;t',t')]\}_a + \{[\theta(t-t')\theta(t-s) \\ & \quad \times R(k;t,s)\theta(t'-s)R(k;t',s)C(k;s,s)]\}_b \\ & \quad - \{[\theta(t-t')\theta(t-s)R(k;t,s)\delta_{t',s}C(k;s,t')]\}_c. \end{aligned} \tag{29}$$

Specializing to the case $t > s > t'$ corresponds to $\{\cdot\}_b = 0$ and $\{\cdot\}_c = 0$, leaving

$$\begin{aligned} & \theta(t-t')R(k;t,t')C(k;t',t') \\ & \quad = [\theta(t-t')\theta(t-s)R(k;t,s)\theta(s-t')R(k;s,t')C(k;t',t')]. \end{aligned} \tag{30}$$

We now use the contraction property of the Heaviside function $\theta(t-s)\theta(s-t') = \theta(t-t')$ to write Eq. (30) as

$$\begin{aligned} & \theta(t-t')\underline{R(k;t,t')}C(k;t',t') \\ & \quad = \theta(t-t')\underline{R(k;t,s)R(k;s,t')} \\ & \qquad \times C(k;t',t'). \end{aligned} \tag{31}$$

[The underlined areas denote the origin of Eq. (32).] From this above result, we can deduce the transitive property of the renormalized response

$$R(k;t,t') = R(k;t,s)R(k;s,t'). \tag{32}$$

These may seem like small results but the fact is that the renormalized perturbation theories of turbulence, which looked so promising initially, have been essentially in a static state for at least three decades. Sometimes the subject is described as being "mired in controversy" but in reality a sober appraisal reveals only a few minor unresolved issues. With the difficulties of using exponential representations of time dependences resolved, the way is clear to explore and learn from the relationships between the different classes of theories. This also includes renormalization group methods, where there is a relationship between DIA [1] and the method of iterative averaging [10,11].

In this short communication we have already shown that our time-ordering approach can be used to prove the transitivity of the renormalized response function. This is not, in itself, a trivial result and moreover has important implications for the dimensional reduction of this type of theory [12]. We have also used these methods to derive a different Langevin equation model of turbulence. With the *ansatz* of local energy transfer to determine the response (see Ref. [13] and references therein), along with an assumption of an exponential relationship between the response function and the eddy damping, as given by Eq. (5), we find

$$\begin{aligned} \left(\frac{\partial}{\partial t} + 2\nu k^2\right) C(k;t) = {} & 2\int d^3j L(\mathbf{k},\mathbf{j}) D(k,j,|\mathbf{k}-\mathbf{j}|;t) C(|\mathbf{k}-\mathbf{j}|;t) \\ & \times [C(j;t) - C(k;t)] \\ = {} & -2\omega(k;t)C(k;t), \end{aligned} \tag{33}$$

$$\begin{aligned} \omega(k;t) = {} & -\int d^3j L(\mathbf{k},\mathbf{j}) D(k,j,|\mathbf{k}-\mathbf{j}|;t) \frac{C(|\mathbf{k}-\mathbf{j}|;t)}{C(k;t)} \\ & \times [C(j;t) - C(k;t)], \end{aligned} \tag{34}$$

and

$$\frac{\partial D(k,j,|\mathbf{k}-\mathbf{j}|;t)}{\partial t} = 1 - [(\nu k^2 + \nu j^2 + \nu|\mathbf{k}-\mathbf{j}|^2) + \omega(k;t) + \omega(j;t) + \omega(|\mathbf{k}-\mathbf{j}|;t)]D(k,j,|\mathbf{k}-\mathbf{j}|;t), \tag{35}$$

where

$$L(\mathbf{k},\mathbf{j}) = -2M_{\alpha\beta\gamma}(\mathbf{k})M_{\beta\alpha\delta}(\mathbf{j})P_{\gamma\delta}(\mathbf{k}-\mathbf{j}). \tag{36}$$

The initial conditions can be taken as

$$C(k;t=0) = \frac{E(k;t=0)}{4\pi k^2}, \tag{37}$$

where $E(k;t=0)$ is some arbitrarily chosen initial energy spectrum, and

$$D(k,j,|\mathbf{k}-\mathbf{j}|;t=0) = 0. \tag{38}$$

This is similar to the test-field model [14], but has an extra term in the equation for the eddy damping. The extra term cancels infrared divergences and this means that (unlike the test-field model) it does not require an additional hypothesis and adjustable constant to be compatible with the Kolmogorov distribution. An account of this work is in preparation.

Finally, in the interests of completeness, we should mention that following the seminal paper of Leith [8] the fundamental issues involved in obtaining fluctuation-dissipation relationships for chaotic systems have received attention, particularly from the point of view of dynamical systems theory [15–19]. It has been shown that a general fluctuation-dissipation relationship exists for systems which are mixing. The precise form of the relationship for any specific system is found to depend on the invariant measure of that system. The relationship of our present work to this existing activity raises several interesting questions and we intend to address these in our forthcoming paper mentioned above.

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